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A probabilistic version of the game of Zombies and Survivors on graphs ☆

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ABSTRACT

We consider a new probabilistic graph searching game played on graphs, inspired by the familiar game of Cops and Robbers. In Zombies and Survivors, a set of zombies attempts to eat a lone survivor loose on a given graph. The zombies randomly choose their initial location, and during the course of the game, move directly toward the survivor. At each round, they move to the neighboring vertex that minimizes the distance to the survivor; if there is more than one such vertex, then they choose one uniformly at random. The survivor attempts to escape from the zombies by moving to a neighboring vertex or staying on his current vertex. The zombies win if eventually one of them eats the survivor by landing on their vertex; otherwise, the survivor wins. The zombie number of a graph is the minimum number of zombies needed to play such that the probability that they win is at least 1/2. We present asymptotic results for the zombie numbers of several graph families, such as cycles, hypercubes, incidence graphs of projective planes, and Cartesian and toroidal grids.

1. Introduction

A number of variants of the popular graph searching game Cops and Robbers have been studied. For example, we may allow a cop to capture the robber from a distance k, where k is a non-negative integer [9,10], play on edges [15], allow the robber to capture the cops [11], allow one or both players to move with different speeds [2,13,16,17] or to teleport, have the cops move one at a time [4,5,26], have the cops play on edges and the robber on vertices [23,28], or make the robber invisible or drunk [20,22,21]. For additional background on Cops and Robbers and its variants, see the book [12] and the surveys [3,6,7].

For a given connected graph G and given $k \in \mathbb{N}$, we consider the following probabilistic variant of Cops and Robbers, which is played over a series of discrete time-steps. In the game of *Zombies and Survivors*, suppose that k *zombies* (akin to the cops) start the game on random vertices of G; each zombie, independently, selects a vertex uniformly at random to start with. Then the *survivor* (akin to the robber) occupies some vertex of G. As zombies have limited intelligence, in each round, a given zombie moves towards the survivor along a shortest path connecting them. In particular, the zombie decreases the distance from its vertex to the survivor's. If there is more than one neighbor of a given zombie that is closer to the survivor

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than the zombie is, then they move to one of these chosen uniformly at random. Each zombie moves independently of all other zombies. As in Cops and Robbers, the survivor may move to another neighboring vertex, or *pass* and not move. The zombies win if one or more of them *eat* the survivor; that is, land on the vertex which the survivor currently occupies. The survivor, as survivors should do in the event of a zombie attack, attempts to survive by applying an optimal strategy; that is, a strategy that minimizes the probability of being captured. Note that there is no strategy for the zombies; they merely move on geodesics towards the survivor in each round. Note that since zombies always move toward the survivor, he can pass at most D times, where D is a diameter of G, before being eaten by some zombie. Note also that the game can be extended to the case of G being disconnected, by having zombies that lie in connected components of G different from that of the survivor simply follow a random walk. Nevertheless, in this paper we will only consider connected graphs. We note also that our probabilistic version of Zombies and Survivors was inspired by a deterministic version of this game (with similar rules, but the zombies may choose their initial positions, and also choose which shortest path to the survivor they will move on) first considered in [18].

Let $s_k(G)$ be the probability that the survivor wins the game, provided that he follows the optimal strategy. Clearly, $s_k(G) = 1$ for $k < c(G)$, where $c(G)$ is the cop number of G. On the other hand, $s_k(G) < 1$ provided that there is a strategy for $k \geq c(G)$ cops in which the cops always try to get closer to the robber, since with positive probability the zombies may follow such a strategy. Usually, $s_k(G) > 0$ for any $k \geq c(G)$; however, there are some examples of graphs for which $s_k(G) = 0$ for every $k \geq c(G)$ (consider, for example, trees). Further, note that $s_k(G)$ is a non-decreasing function of k (that is, for every $k \geq 1$, $s_{k+1}(G) \leq s_k(G)$), and $s_k(G) \to 0$ as $k \to \infty$. The latter limit follows since the probability that each vertex is initially occupied by at least one zombie tends to 1 as $k \to \infty$.

Define the *zombie number* of a graph G by

$$z(G) = \min\{k \geq c(G) : s_k(G) \leq 1/2\}.$$

This parameter is well defined since $\lim_{k\to\infty} s_k(G) = 0$. In other words, $z(G)$ is the minimum number of zombies such that the probability that they eat the survivor is at least 1/2. The ratio $Z(G) = z(G)/c(G) \geq 1$ is the *cost of being undead*. Note that there are examples of families of graphs for which there is no cost of being undead; that is, $Z(G) = 1$ (as is the case if G is a tree), and, as we show in the next section, there are examples of graphs with $Z(G) = \Theta(n)$.

The paper is organized as follows. In Section 2, we give an example of a sequence of graphs $(G_n)_{n\in\mathbb{N}}$ having $Z(G_n) = \Theta(n)$. In Section 3, we discuss cycle graphs. Theorem 3.3 gives the asymptotic value of the zombie number of cycles. In Section 4, we consider the zombie number of the incidence graphs of projective planes. By using double exposure and coupon collector problems, we show in Theorem 4.1 that about two times more zombies are needed to eat the survivor than cops. We consider hypercubes Q_n in Section 5, and show in Theorem 5.1 that $z(Q_n) \sim \frac{2}{3}n$, as $n \to \infty$. The final section considers both Cartesian grids and grids formed by products of cycles (so called *toroidal grids*). In toroidal grids, we prove in Theorem 6.2 a lower bound for the zombie number of $\sqrt{n}/(\omega \log n)$, where $\omega = \omega(n)$ is going to infinity as $n \to \infty$. The proof relies on the careful analysis of a strategy for the survivor.

Throughout, we will use the following version of *Chernoff's bound*. For more details, see, for example, [19]. Suppose that $X \in \mathrm{Bin}(n, p)$ is a binomial random variable with expectation $\mu = np$. If $0 < \delta < 1$, then

$$\mathbb{P}[X < (1-\delta)\mu] \leq \exp\left(-\frac{\delta^2\mu}{2}\right),$$

and if $\delta > 0$,

$$\mathbb{P}[X > (1+\delta)\mu] \leq \exp\left(-\frac{\delta^2\mu}{2+\delta}\right).$$

The above bounds show that with high probability X cannot be too far away from its expectation. However, it is also true that with high probability X cannot be too close to $\mathbb{E}[X]$. We will use this fact only for the $p = 1/2$ case. Let $X \in \mathrm{Bin}(n, 1/2)$. First, let us use Stirling's formula ($k! \sim \sqrt{2\pi k}(k/e)^k$) and observe that for each t such that $0 \leq t \leq n$ we have

$$\mathbb{P}[X = t] \leq \mathbb{P}\Big[X = \lfloor n/2 \rfloor\Big] = \frac{\binom{n}{\lfloor n/2\rfloor}}{2^n} \sim \sqrt{\frac{2}{\pi n}} < \frac{1}{\sqrt{n}}.$$

Hence, for each $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$ (for example, let $c = \varepsilon/3$) such that

$$\mathbb{P}\Big[|X - n/2| < c\sqrt{n}\Big] < \varepsilon. \tag{1}$$

For a reference on graph theory the reader is directed to [29]. For graphs G and H, define the *Cartesian product* of G and H, written $G\square H$, to have vertices $V(G) \times V(H)$, and vertices (a, b) and (c, d) are joined if $a = c$ and $bd \in E(H)$ or $ac \in E(G)$ and $b = d$. Many results in the paper are asymptotic in nature as $n \to \infty$. We emphasize that the notations $o(\cdot)$ and $O(\cdot)$ refer to functions of n, not necessarily positive, whose growth is bounded. We say that an event in a probability space holds *asymptotically almost surely* (or *a.a.s.*) if the probability that it holds tends to 1 as n goes to infinity. Finally, for simplicity we will write $f(n) \sim g(n)$ if $f(n)/g(n) \to 1$ as $n \to \infty$; that is, when $f(n) = (1 + o(1))g(n)$.

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