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Pursuit evasion on infinite graphs

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ABSTRACT

The cop-and-robber game is a game between two players, where one tries to catch the other by moving along the edges of a graph. It is well known that on a finite graph the cop has a winning strategy if and only if the graph is constructible and that finiteness is necessary for this result.

We propose the notion of weakly cop-win graphs, a winning criterion for infinite graphs which could lead to a generalisation. In fact, we generalise one half of the result, that is, we prove that every constructible graph is weakly cop-win. We also show that a similar notion studied by Chastand et al. (which they also dubbed weakly cop-win) is not sufficient to generalise the above result to infinite graphs.

In the locally finite case we characterise the constructible graphs as the graphs for which the cop has a so-called protective strategy and prove that the existence of such a strategy implies constructibility even for non-locally finite graphs.

1. Introduction

The game of cops and robbers was first studied by Nowakowski and Winkler [10] and Quilliot [14,15]. Since its first appearance numerous variants and aspects of the game have been studied. The book [3] by Bonato and Nowakowski gives a fairly good overview.

The game is played on the vertex set of a graph between two players called the cop and the robber. Both players have perfect information. They alternately take turns, with the cop going first. In the first round their move is to select a starting vertex. In each consecutive round they can either move to a neighbour of the vertex they are at, or stay where they are.

The goal of the cop is to "catch" the robber, that is, to occupy the same vertex as the robber after finitely many steps. The robber wins if he can avoid being caught forever. Since one of the two events must happen, von Neumann's Theorem implies the existence of a winning strategy for one of the two players.

While the focus has been mostly on finite graphs there have also been several publications treating the infinite case [1,2,6,8,11–13].

One of the reasons why infinite graphs have not received more attention is that many very basic results break down as soon as we leave the realm of finite graphs. The most striking example of this is probably that graphs which contain an isometric copy of an infinite path can never be cop-win. Even worse, the game cannot be won by any finite number of cops on such a graph. The robber's strategy would simply consist of starting "further out" along the path than any cop and then running away in a straight line. Clearly the cops can never catch up and hence the robber can avoid being captured forever.

But even if we remove this obvious obstruction, results fail to generalise. For example, Hahn et al. [8] showed that there are infinite chordal graphs of diameter 2 which are not cop-win. This contrasts the fact that finite chordal graphs are always cop-win.

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In this paper we study an altered winning criterion which seems to be better adapted to infinite graphs. It coincides with the original winning criterion for finite graphs, but allows to generalise some results to infinite graphs. We only consider the most basic version of the game where one cop tries to catch one robber, but it is certainly possible that a similar approach works for more general variants of the game as well.

The following necessary and sufficient condition for the existence of a winning strategy for the cop on a finite graph was discovered independently by Nowakowski and Winkler [10] and Quilliot [14]. The main motivation for the present paper is that there does not seem to be a satisfactory generalisation of this result to infinite graphs.

Theorem 1 *(Nowakowski and Winkler [10], Quilliot [14]). A finite graph is cop-win if and only if it is constructible.*

Constructible here means that G can be constructed according to certain rules which will be explained in the next section. The characterisation does not remain valid for infinite graphs. One reason for this is, as mentioned, graphs with an isometric copy of an infinite path can never be cop-win.

While this problem was already addressed by Chastand et al. [6], their proposed solution turns out to be unsatisfactory. They introduced the notion of C-weakly cop-win graphs[2] where the cop wins the game if he can either catch the robber or chase him away. They proved that certain infinite constructible graphs are cop-win in this sense and asked whether the C-weakly cop-win graphs are exactly the constructible graphs.

In this paper we show that this is not the case if we use their exact definition. With the following slightly modified definition of weakly cop-win based on the same intuition we are able to prove that the cop has a winning strategy on every constructible graph: call a graph weakly cop win if there is a strategy for the cop which prevents the robber from visiting any vertex infinitely many times.

We also introduce protective strategies, which are a special kind of winning strategies and show that every graph which admits such a protective strategy is constructible. For locally finite graphs we can even show that being constructible is equivalent to the existence of a protective strategy.

Finally we investigate dismantable graphs (which in the finite case are exactly the constructible graphs) and give a sufficient condition for an infinite dismantable graph to be weakly cop-win.

The rest of this paper is structured as follows. After introducing some basic notions, we outline the rules of the game and give a proof of Theorem 1. We briefly discuss, why a similar characterisation is not possible for infinite cop-win graphs. In Section 5 we outline the approach of Chastand et al. and show that there is a locally finite constructible graph which is not C-weakly cop-win. We then proceed to introduce our modified definition of weakly cop-win graphs and prove the results mentioned above.

The main result of Section 7, and probably of the whole paper is Theorem 9 which states that every constructible graph is weakly cop-win. In Section 8 we introduce the protective strategies mentioned earlier and show that a locally finite graph is constructible if and only if it admits a protective strategy. Section 9 contains some results on dismantable graphs. We conclude the paper with some interesting open questions about weakly cop-win graphs.

2. Basic notions

Throughout this paper $G = (V, E)$ will denote a simple graph G with vertex set V and edge set E. Graph theoretical notions which are not explicitly defined will be taken from [7].

Since throughout most of the paper the vertex sets of the graphs in consideration are well ordered, we start by recalling some facts about well orders and ordinal numbers. All of those facts can be found in most standard text books on set theory, readers not familiar with ordinals and well orders see for example [16] for a more detailed introduction.

A *well order* of a set is a total order (i.e. any two elements are comparable) in which every subset has a minimal element.

One elementary fact about well orders is that in such an order there is no infinite descending chain, that is, every sequence $x_1 < x_2 < x_3 < \dots$ can only have finitely many elements.

It is also known that every well order is order isomorphic to some ordinal number where an *order isomorphism* is a bijective, order preserving function whose inverse is also order preserving. We denote by Ord the class of ordinal numbers. The ordinals themselves are well ordered with respect to being an initial piece of one another. This order is such that we can identify α (and thus also every well ordered set which is order isomorphic to α) with the set $\{\nu \in \mathrm{Ord} \mid \nu < \alpha\}$ by means of an order isomorphism.

Every ordinal α has a successor which is obtained from α by adding a new element which is larger than all other elements. We call β a *successor ordinal* if it is the successor of some $\alpha \in \mathrm{Ord}$ and write $\beta = \alpha + 1$ and $\alpha = \beta - 1$, respectively. If an ordinal is not a successor we call it a *limit ordinal*.

We identify the (up to order isomorphism) unique well order on an n-element set with the natural number $n \in \mathbb{N}$. The set $\mathbb{N}$ itself is also well ordered and can be identified with the smallest infinite ordinal number. If an order is order isomorphic to $\alpha \leq \mathbb{N}$ we call it a *natural order*.

[2] Of course, Chastand et al. call them weakly cop-win. The reason we call them C-weakly cop-win is that we would like to reserve the term weakly cop-win for the new winning criterion introduced in Section 6.