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Algorithm on rainbow connection for maximal outerplanar graphs ☆

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ABSTRACT

In this paper, we consider rainbow connection number of maximal outerplanar graphs (MOPs) on algorithmic aspect. For the (MOP) G, we give sufficient conditions to guarantee that $rc(G) = diam(G)$. Moreover, we produce the graph with given diameter D and give their rainbow coloring in linear time. X. Deng et al. [4] give a polynomial time algorithm to compute the rainbow connection number of MOPs by the Maximal fan partition method, but only obtain a compact upper bound. J. Lauri [11] proved that, for chordal outerplanar graphs given an edge-coloring, to verify whether it is rainbow connected is NP-complete under the coloring, it is so for MOPs. Therefore we construct Central-cut-spine of MOP G, by which we design an algorithm to give a rainbow edge coloring with at most $2rad(G) + 2 + c, 0 \le c \le rad(G) - 2$ colors in polynomial time.

1. Introduction

Graphs considered are finite, simple and connected in this paper. Notations and terminologies not defined here, see West [18]. The concept of rainbow conception was introduced by Chartrand, Johns, McKeon and Zhang in 2008 [9]. Let G be a nontrivial finite simple connected graph on which is assigned a coloring $c : E(G) \to \{1, 2, \cdots, n\}$, $n \in \mathbb{N}$, where adjacent edges may have same color. A *r*ainbow path in G is a path with different colors on it. If for any two vertices of G, there is a rainbow path connecting them, then G is called *r*ainbow connected and c is called a *r*ainbow coloring. Obviously, any graph G has a trivial rainbow coloring by coloring each edge with different colors. Chartrand et al. [9] defined the *r*ainbow connection number $rc(G)$ of graph G as the smallest number of colors needed to make G rainbow connected. For any two vertices u and v in G, the length of a shortest path between them is their distance, denoted by $d(u, v)$. The eccentricity of a vertex v is $ecc(v) := max_{x \in V(G)} d(v, x)$. The diameter of G is $diam(G) := max_{x \in V(G)} ecc(x)$. The radius of G is $rad(G) := min_{x \in V(G)} ecc(x)$. Distance between a vertex v and a set $S \subseteq V(G)$ is $d(v, S) := min_{x \in S} d(v, x)$. The k-step open neighborhood of a set $S \subseteq V(G)$ is $N_k(S) := \{x \in V(G) | d(x, S) = k\}$, $k \in \{0, 1, 2, \ldots\}$. The degree of a vertex v is $degree(v) :=| N_1(v) |$. The maximum degree of G is $\Delta(G) := max_{x \in V(G)} degree(x)$. The girth of a graph G is $g(G) :=$ the length of maximal induced cycle in G. A vertex is called pendant if its degree is 1. Let $n(G) =| V(G) |$ and $e(G)$ be the

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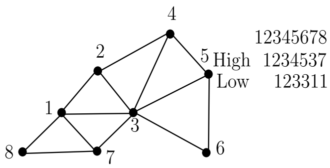

Fig. 1. Example.

size of G. Obviously $diam(G) \leq rc(G) \leq e(G)$. From [9], we know that rainbow connection number of any complete graph is 1 and that of a tree is its size.

Obviously, we know that cut-edges must have distinct colors when G is rainbow connected. Thus stars have arbitrarily large rainbow connection number while having diameter 2. Therefore, it is significant to seek upper bound on $rc(G)$ in terms of $diam(G)$ in 2-edge-connected graphs. Chandran et al. [8] showed that $rc(G) \leq rad(G)(rad(G)+2)$ when G is 2-edge-connected, and hence $rc(G) \leq diam(G)(diam(G)+2)$. Li et al. [13] proved that $rc(G) \leq 5$ when G is a 2-edge-connected graph with diameter 2. Li et al. [14] proved that $rc(G) \leq 9$ when G is a 2-edge-connected graph with diameter 3.

Recalling an outerplanar graph is a planar graph which has a plane embedding with all vertices placed on the boundary of a face, usually taken to be the exterior one. A MOP is an outerplanar graph which can not be added any line without losing outerplanarity.

By [1], a MOP can be recursively defined as follows: **(a)** K_3 is a MOP. **(b)** For a MOP H_1 embedded in the plane with vertices lying in the exterior face F_1, H_2 is obtained by joining a new vertex to two adjacent vertices on F_1. Then H_2 is a MOP. (**c**) Any MOP can be constructed by finite steps of **(a)** and **(b)**.

Note each inner face of a MOP H is a triangle and the connectivity $\kappa(H) = 2$. Moreover, H can be represented by two line arrays $High(1), High(2), \cdots, High(n)$ and $Low(1), Low(2), \cdots, Low(n)$. Here for any vertex i, $High(i)$ and $Low(i)$ are labels of its two neighbors whose labels are less than i, and $High(i) > Low(i)$; and $High(1), Low(1)$ and $Low(2)$ are undefined, and $High(2) = 1$. Fig. 1 illustrates a MOP and its canonical representation.

Property (A). A graph is outerplanar if and only if it has no K_4 or $K_{2,3}$ minor.

We summarize some results for the rainbow connection number of graphs in the following.

Huang et al. proved that if G is a bridgeless outerplanar graph of order n and $diam(G) = 2$, then $rc(G) \leq 3$ and the bound is tight. Moreover they proved that if $diam(G) = 3$, then $rc(G) \leq 6$, in [16].

Theorem 1.1. *([9]) For a cycle C_n, we have*

$$rc(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even}, \\ \lceil \frac{n}{2} \rceil & \text{if } n \text{ is odd}. \end{cases}$$

Chandran et al. [8] studied the relation between rainbow connection numbers and connected dominating sets, and they obtained the following results:

(1) *For any bridgeless chordal graph G, $rc(G) \leq 3rad(G)$. Moreover, the result is tight.*
(2) *For any unite interval graph G with $\delta(G) \geq 2$, $rc(G) = diam(G)$.*

A finite simple connected graph G is called a Fan *if it is $P_n \vee K_1$ (the join of P_n and K_1), denoted by Fan_n, for some $n \in \mathbb{N} \setminus \{1\}$. Here the vertex v of K_1 is called central vertex, the edges $v_i v_{i+1} (1 \leq i \leq n-1)$ of $P_n = (v_1 v_2 \cdots v_n)$ are called path edges, and the edges $v_i v$ between P_n and K_1 are called spoke edges.*

Theorem 1.2. *([4]) The rainbow connection number of Fan_n satisfies*

$$rc(Fan_n) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } 3 \leq n \leq 6, \\ 3 & \text{if } n \geq 7. \end{cases}$$

Theorem 1.3. *([5]) Let G be a bridgeless outerplanar graph of order n.*

1. *If $diam(G) = 2$, then*

$$rc(G) = \begin{cases} 3 & \text{if } G = F_n \ (n \geq 7) \text{ or } C_5, \\ 2 & \text{otherwise}. \end{cases}$$

2. *If $diam(G) = 3$, then $3 \leq rc(G) \leq 4$ and the bound is tight.*