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The security number of strong grid-like graphs ☆

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A B S T R A C T

The concept of a secure set in graphs was first introduced by Brigham et al. in 2007 as a generalization of defensive alliances in graphs. Defensive alliances are related to the defense of a single vertex. However, in a general realistic settings, a defensive alliance should be formed so that any attack on the entire alliance or any subset of the alliance can be defended. In this sense, secure sets represent an attempt to develop a model of this situation. Given a graph $G = (V, E)$ and a set of vertices $S \subseteq V$ of G, the set S is a secure set if it can defend every attack of vertices outside of S, according to an appropriate definition of "attack" and "defense". The minimum cardinality of a secure set in G is the security number $s(G)$. In this article we obtain the security number of grid-like graphs, which are the strong products of paths and cycles (grids, cylinders and toruses). Specifically we show that for any two positive integers $m, n \geq 4$, $s(P_m \boxtimes P_n) = \min\{m, n, 8\}$, $s(P_m \boxtimes C_n) = \min\{2m, n, 16\}$ and $s(C_m \boxtimes C_n) = \min\{2m, 2n, 32\}$.

1. Introduction

Secure sets and security number in graphs were first described by Brigham et al. [1] while attempting to improve the well-known concepts of defensive alliances and defensive alliance number in graphs [2]. After this seminal work on secure sets in graphs, relatively few articles have been published regarding this topic. Some general results on security number are presented in [3,4]. According to the definition in [2], defensive alliances only defend a single vertex at a given time. Nevertheless, in general models, a more efficient defensive alliance should be able to defend any attack on the entire alliance or any part of it. Studies of security in product graphs were initiated in [1], and afterwards continued in [5–7] where several bounds and closed formulas on the security number of some grid-like graphs were given. Secure sets have also been investigated in [8–10].

We begin with some notation and terminology. In this paper $G = (V, E)$ denotes a simple graph of order n, minimum degree δ and maximum degree Δ. For a nonempty subset $W \subseteq V$ and any vertex $v \in V$, $N_W(v)$ is the set of neighbors of the vertex v in W, $N_W(v) = \{u \in W : uv \in E(G)\}$, and $\delta_W(v) = |N_W(v)|$ denotes the degree of v in W. If $W = V$, then we use the notation $N(v)$ and call it the *open neighborhood* of v. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *open*

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neighborhood of a set W is $N(W)=\bigcup_{v\in W}N(v)$ and the *closed neighborhood* of W is $N[W]=N(W)\cup W$. The subgraph induced by a set W is denoted by $\langle W\rangle$, and the complement of W is denoted by $\overline{W}$.

The definition of a secure set is based on the following rules. Consider a set of vertices S. A vertex $y\in N[S]-S$ can attack only one neighbor in S (it does not matter if y is adjacent to several vertices in S). On the other hand, a vertex $x\in S$ can defend only one vertex in $N[x]\cap S$. We now present a formal definition of secure sets according to [1].

- For any $S=\{v_1,v_2,\dots,v_r\}\subseteq V$, an attack on S is formed by any r mutually disjoint sets $A=\{A_1,A_2,\dots,A_r\}$, for which $A_i\subseteq N_{\overline{S}}(v_i)$, $1\le i\le r$.
- A defense of S is formed by any r mutually disjoint sets $D=\{D_1,D_2,\dots,D_r\}$ for which $D_i\subseteq N_S[v_i]$, $1\le i\le r$.
- Attack A is defendable if there exists a defense D such that $|D_i|\ge|A_i|$ for $1\le i\le r$.
- Set S is secure if and only if every attack on S is defendable.

The minimum cardinality of a secure set in a graph G is the *security number* and is denoted by $s(G)$. A secure set S of cardinality $s(G)$ is called a $s(G)$-set. Throughout the article we use the following characterization of secure sets.

Theorem 1 *([1]). A set S is a secure set in a graph G if and only if for every $X\subseteq S$, $|N[X]\cap S|\ge|N[X]-S|$.*

From now on we call the expression $|N[X]\cap S|\ge|N[X]-S|$ the *security condition* for X. Studies on the security number in product graphs were initiated in [1], where the authors gave upper bounds for the Cartesian product of paths and cycles. Moreover, to prove the equality in these bounds was left as an open problem, which was solved in [5]. There was proved that for any integers $m,n\ge4$, it follows that $s(P_m\square P_n)=\min\{m,n,3\}$, $s(P_m\square C_n)=\min\{2m,n,6\}$ and $s(C_m\square C_n)=\min\{2m,2n,12\}$. Other studies of the global security number of the Cartesian product of paths and cycles where presented in [6,7]. In the present article we prove a formula for the security number of the strong product of paths and/or cycles.

We recall that the strong product of two graphs $G=(U,E_1)$ and $H=(V,E_2)$ is the graph $G\boxtimes H$, with the vertex set $\{(a,b):a\in U,\,b\in V\}$ and two vertices (a,b) and (c,d) of $U\times V$ are adjacent in $G\boxtimes H$ if and only if, either ($a=c$ and $bd\in E_2$), ($b=d$ and $ac\in E_1$), or ($ac\in E_1$ and $bd\in E_2$). The graphs G and H are called the *factors* of the product. For a vertex $a\in U$, the set of vertices $\{(a,b):b\in V\}$ is called an H-*layer* and is denoted by ${}^aH$. Similarly, for a vertex $b\in V$, the set of vertices $\{(a,b):a\in U\}$ is called a G-*layer* and is denoted by G^b. It is clear that the graph induced by any G-layer is isomorphic to G and, analogously, the graph induced by any H-layer is isomorphic to H. The projection of a set $W\subset U\times V$ onto G is defined by $P_G(W)=\{u\in U:(u,v)\in W\}$. Analogously, the projection of W onto H is $P_H(W)=\{v\in V:(u,v)\in W\}$. The proof of the following result is the main goal of this article.

Theorem 2. *Let $m,n\ge2$ be two integers. Then*

(i) $s(P_m\boxtimes P_n)=\min\{m,n,8\}$,
(ii) $s(P_m\boxtimes C_n)=\min\{2m,n,16\}$,
(iii) $s(C_m\boxtimes C_n)=\begin{cases}5, & \text{if } m=n=3,\\ \min\{2m,2n,32\}, & \text{otherwise.}\end{cases}$

We split the proof of the results above in two parts (see Sections 2 and 3). Notice that the graph $C_3\boxtimes C_3$ is isomorphic to the complete graph K_9, and from [1] we know that $s(C_3\boxtimes C_3)=5$.

The paper is organized as follows. Throughout the article $U=\{u_0,\dots,u_{m-1}\}$ and $V=\{v_0,\dots,v_{n-1}\}$ represent the vertex sets of graphs G and H of order m and n, respectively, where G and H are a path or a cycle. All the operations with the subscripts are done modulo m or n, respectively, for those cases in which the corresponding factor graph is a cycle. We only consider non-symmetrical cases.

2. Proofs of the upper bounds

Let S be a set of vertices in a graph G and let $\mathcal{UD}$ be a set of disjoint pairs of vertices $\{u,v\}$ such that $u\in S$ and $v\in N[S]-S$. The set $\mathcal{UD}$ is a *universal defense* for S, if for any attack on S, the attack of a vertex $v\in N[S]-S$ can be repelled by a vertex $u\in S$ such that $\{u,v\}\in\mathcal{UD}$.

Lemma 3. *Let $G=(V,E)$ be a graph and let $S\subset V$. If there exists a universal defense for S, then S is a secure set.*

Proof. Let $A=\{A_1,A_2,\dots,A_r\}$ be an attack on S and let $A_i=\{v_{i1},v_{i2},\dots,v_{ik_i}\}$ for every $i\in\{1,\dots,r\}$. Since there exists a universal defense $\mathcal{UD}$ for S, every v_{ij} belongs to a pair $\{u_{ij},v_{ij}\}\in\mathcal{UD}$, where $u_{ij}\in S$ and the vertex u_{ij} does not appear in any other pair. Thus, the set $D_i=\{u_{i1},u_{i2},\dots,u_{ik_i}\}$ satisfies the condition $|D_i|\ge|A_i|$ for every $i\in\{1,\dots,r\}$. Therefore, S is a secure set. □

In the proof of the following claims we construct a universal defense for the corresponding set and then we use the lemma above.