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Stable matchings of teachers to schools ☆

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A B S T R A C T

Several countries successfully use centralized matching schemes for school or higher education assignment, or for entry-level labour markets. In this paper we explore the computational aspects of a possible similar scheme for assigning teachers to schools. Our model is motivated by a particular characteristic of the education system in many countries where each teacher specializes in two subjects. We seek stable matchings, which ensure that no teacher and school have the incentive to deviate from their assignments. Indeed we propose two stability definitions depending on the precise format of schools' preferences. If the schools' ranking of applicants is independent of their subjects of specialism, we show that the problem of deciding whether a stable matching exists is NP-complete, even if there are only three subjects, unless there are master lists of applicants or of schools. By contrast, if the schools may order applicants differently in each of their specialization subjects, the problem of deciding whether a stable matching exists is NP-complete even in the presence of subject-specific master lists plus a master list of schools. Finally, we prove a strong inapproximability result for the problem of finding a matching with the minimum number of blocking pairs with respect to both stability definitions.

1. Introduction

In the organization of education, several countries or regions use various centralized schemes to allocate children to public schools (e.g., in Boston and New York [1,2]), students to universities (e.g., in Hungary [6]), and intending junior doctors to training positions in hospitals (e.g., in the USA [21]), etc. These schemes are usually not dictatorial in the sense that they take into account the wishes of both sides of the market: students may express their preferences over the universities they wish to attend, and the universities may order their applicants based on some kind of evaluation. After analyzing several successful and unsuccessful schemes Roth [17,18] convincingly argued that a crucial property for success is so-called *stability*, introduced in the seminal paper by Gale and Shapley [10]. Stability means that no unmatched student–school pair

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should simultaneously prefer each other to their current assignee(s) (if any). In many real markets, each instance not only admits a stable matching, but it is also possible to find such a matching efficiently.

However, sometimes there are circumstances leading to additional structural requirements. For example, married couples may wish to be allocated to the same hospital or at least to hospitals that are geographically close [15,7], or schools may wish to have the right to close a study programme if the number of applicants does not meet a certain lower quota [6]. In such cases, a suitable notion of stability has to be defined that really mirrors the intentions of the participants and motivates them to obey the recommended assignment. Alas, a stable matching is not necessarily bound to exist; and even worse, it is often a computationally difficult problem to decide whether in the given situation one does exist [16].

The topic of this paper is motivated by the problems arising in the labour market for teachers. Traditionally, a teacher for the upper elementary or lower secondary level of education in Slovakia and the Czech republic (and in fact in many other countries and regions, such as Germany [4] and Flanders [9]) specializes in two curricular domains (from now on called *subjects*), e.g., Mathematics and Physics, Chemistry and Biology, or Slovak language and English etc. When a school is looking for new teachers, it may have a limited number of lessons to cover (or teaching hours to fill) in each subject. Thus we suppose that each school has different capacity for each subject and that it will be willing to employ a set of teachers in such a way that these capacities will not be exceeded. Cechlárová et al. [8] studied a variant of this problem where the trainee teachers could only express which schools are acceptable for them, and which are not, without ordering them according to their preferences, and the schools had no input. In these settings, the aim was to assign as many trainee teachers as possible (ideally all of them) by respecting the schools' capacities.

The aim of this paper is to study algorithmic aspects of the problem of assigning teachers to schools within the framework of two-sided preferences. We suppose that teachers rank in order of preference their acceptable schools according to their own criteria, and vice versa, schools rank-order their applicants similarly [14]. In this context we suggest two stability definitions and study the computational complexity of problems concerned with finding stable matchings (or reporting that none exist). These definitions and the associated complexity results depend on the nature of the schools' preference lists.

The main results and the organization of the rest of the paper are as follows. In Section 2 we introduce relevant technical concepts and illustrate them by means of simple examples. In Section 3 we deal with the case when each school has a linear ordering on the set of teachers who apply for a position. We show that in this general case the problem of deciding whether a stable matching exists is NP-complete, even if there are only three subjects in total. This result is perhaps not unexpected, since the problem studied in this paper bears some resemblance to the Hospitals/Residents problem with Couples (HRC), and the problem of deciding whether a given instance of HRC admits a stable matching is NP-complete [16].

By contrast, we show in Section 4 that if either the preferences of schools are derived from a common master list of teachers, or vice versa if the preferences of teachers are derived from a common master list of schools, a unique stable matching exists and it can be found using straightforward extensions of the classical Serial Dictatorship mechanism [19]. Moreover, the problems with master lists are efficiently solvable without any restrictions on the number of subjects. In Section 5 we modify the stability definition to enable the schools to order the teachers differently according to their two specialization subjects. We show that in this case, the problem of deciding whether a stable matching exists is NP-complete even if there are only three subjects, there are master lists for each subject and there is also a master list of schools. Finally, problems involving finding matchings with the minimum number of blocking pairs are discussed in Section 6, where we show that, with respect to both stability definitions, the problem of finding a matching with the minimum number of blocking pairs is very difficult to approximate.

2. Preliminary definitions and observations

An instance of the Teachers Assignment Problem, TAP for short, involves a set A of applicants (teachers), a set S of schools and a set P of subjects. For ease of exposition, elements of the set P will sometimes be referred to by letters like M, F or I to remind the reader of real subjects taught at schools, such as Mathematics, Physics, or Informatics, etc.

Each applicant $a \in A$ is characterized by a pair of distinct subjects $\mathbf{p}(a) \subseteq P$, where $\mathbf{p}(a) = \{p_1(a), p_2(a)\}$, that define her *type*. Sometimes we shall also say that a particular applicant is of type FM, IM, or FI, etc. Corresponding to each applicant $a \in A$ there is a set $S(a) \subseteq S$ of schools that a finds *acceptable*. Moreover applicant a ranks $S(a)$ in strict order of preference.

Each school $s \in S$ has a certain capacity for each subject: the vector of capacities will be denoted by $\mathbf{c}(s) = (c_1(s), \ldots, c_k(s)) \in \mathbb{N}^k$, where $k = |P|$, and an entry of $\mathbf{c}(s)$ will be called the *partial capacity* of school s. Here, $c_i(s)$ is the maximum number of applicants, whose specialization involves subject p_i, that school s is able to take. Further, each school ranks its applicants in strict order of preference.

Let $S(A) = \{(a, s) : a \in A \wedge s \in S(a)\}$ denote the set of *acceptable* applicant–school pairs. An *assignment* $\mathcal{M}$ is a subset of $S(A)$ such that each applicant $a \in A$ is a member of at most one pair in $\mathcal{M}$. We shall write $\mathcal{M}(a) = s$ if $(a, s) \in \mathcal{M}$ and say that applicant a *is assigned to* school s, and write $\mathcal{M}(a) = \emptyset$ if there is no $s \in S$ with $(a, s) \in \mathcal{M}$. The set of applicants assigned to a school s will be denoted by $\mathcal{M}(s) = \{a \in A : (a, s) \in \mathcal{M}\}$. We shall also denote by $\mathcal{M}_p(s)$ the set of applicants assigned to s whose specialization includes subject p and by $\mathcal{M}_{p,r}(s)$ the set of applicants assigned to s whose specialization is exactly the pair $\{p, r\}$. More precisely,

$$\mathcal{M}_p(s) = \{a \in A : (a, s) \in \mathcal{M} \ \wedge \ p \in \mathbf{p}(a)\}$$