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Embedded connectivity of recursive networks ☆

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ABSTRACT

Let G_n be an n-dimensional recursive network. The h-embedded connectivity $\zeta_h(G_n)$ (resp. edge-connectivity $\eta_h(G_n)$) of G_n is the minimum number of vertices (resp. edges) whose removal results in disconnected and each vertex is contained in an h-dimensional subnetwork G_h. This paper determines ζ_h and η_h for the hypercube Q_n and the star graph S_n, and η_3 for the bubble-sort network B_n.

1. Introduction

It is well known that interconnection networks play an important role in parallel computing/communication systems. An interconnection network can be modeled by a graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network.

The *connectivity* $\kappa(G)$ (resp. *edge-connectivity* $\lambda(G)$) of G is defined as the minimum number of vertices (resp. edges) whose removal from G results in a disconnected graph. The connectivity $\kappa(G)$ and edge-connectivity $\lambda(G)$ of a graph G are two important measurements for fault tolerance of the network since the larger $\kappa(G)$ or $\lambda(G)$ is, the more reliable the network is.

However, the definitions of $\kappa(G)$ and $\lambda(G)$ are implicitly assumed that any subset of system components is equally likely to be faulty simultaneously, which may not be true in real applications, thus they underestimate the reliability of the network. To overcome such a shortcoming, Harary [2] introduced the concept of conditional connectivity by appending some requirements on connected components, Latifi et al. [3] specified requirements and proposed the concept of the restricted h-connectivity. These parameters can measure fault tolerance of an interconnection network more accurately than the classical connectivity. The concepts stated here are slightly different from theirs.

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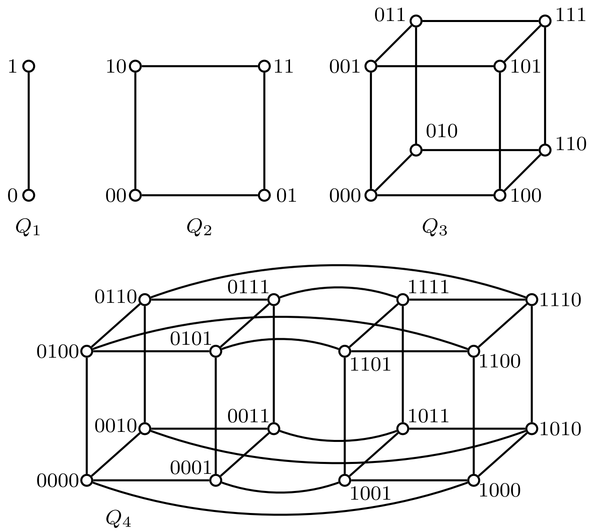

Fig. 1. The n-cubes Q_1, Q_2, Q_3 and Q_4.

For a graph G, $\delta(G)$ denotes its minimum degree. A subset $S \subset V(G)$ (resp. $F \subset E(G)$) is called an *h-vertex-cut* (resp. *h-edge-cut*), if $G - S$ (resp. $G - F$) is disconnected and $\delta(G - S) \geq h$. The *h-super connectivity* $\kappa^h(G)$ (resp. *h-super edge-connectivity* $\lambda^h(G)$) of G is defined as the cardinality of a minimum h-vertex-cut (resp. h-edge-cut) of G.

For any graph G and any integer h, determining $\kappa^h(G)$ and $\lambda^h(G)$ is quite difficult, no polynomial algorithm to compute them has been yet known so far. In fact, the existence of $\kappa^h(G)$ and $\lambda^h(G)$ is an open problem for $h \geq 1$. Only a little knowledge of results has been known on κ^h and λ^h for some special classes of graphs for any h, such as the hypercube Q_n and the star graph S_n.

In order to facilitate the expansion of the network, and to use the same routing algorithm or maintenance strategy as used in the original one, large-scale parallel computing systems always take some networks of recursive structures as underlying topologies, such as the hypercube Q_n, the star graph S_n, the bubble-sort graph B_n and so on. Since the presence of vertex and/or edge failures maybe disconnects the entire network, one hopes that every remaining component has undamaged subnetworks (i.e., smaller networks with same topological properties as the original one). Under this consideration, Yang et al. [12] proposed the concept of embedded connectivity.

Let G_n be an n-dimensional recursive network. For a positive integer h with $h \leq n-1$, there is a sub-network $G_h \subset G_n$. Let $\delta_h = \delta(G_h)$.

A subset $F \subset V(G_n)$ (resp. $F \subset E(G_n)$) is an *h-embedded vertex-cut* (resp. *h-embedded edge-cut*) if $G_n - F$ is disconnected and each vertex is contained in an h-dimensional subnetwork G_h. The *h-embedded connectivity* $\zeta_h(G_n)$ (resp. *edge-connectivity* $\eta_h(G_n)$) of G_n is defined as the cardinality of a minimum h-embedded vertex-cut (resp. h-embedded edge-cut) of G_n.

By definition, if S is an h-embedded vertex-cut of G_n with $|S| = \zeta_h(G_n)$, then $G_n - S$ contains a sub-network G_h, and so $\delta(G_n - S) \geq \delta_h$, which implies that S is a δ_h-vertex-cut of G_n. Thus, $\kappa^{\delta_h}(G_n) \leq |S| = \zeta_h(G_n)$. Similarly, $\lambda^{\delta_h}(G_n) \leq \eta_h(G_n)$. These facts are useful and we write them as the following lemma.

Lemma 1.1. *For $h \leq n-1$, $\zeta_h(G_n) \geq \kappa^{\delta_h}(G_n)$ if $\zeta_h(G_n)$ exists, and $\eta_h(G_n) \geq \lambda^{\delta_h}(G_n)$ if $\eta_h(G_n)$ exists.*

Using Lemma 1.1, for a star graph S_n and a bubble-sort graph B_n, Yang et al. [12,13] determined $\zeta_2(S_n) = 2n-4$ for $n \geq 3$, $\eta_2(S_n) = 2n-4$ for $n \geq 3$ and $\eta_3(S_n) = 6(n-3)$ for $n \geq 4$; and $\zeta_2(B_n) = 2n-4$ for $n \geq 3$. In this paper, we will determine ζ_h and η_h for Q_n and S_n for any $h \leq n-1$ and determine $\eta_3(B_n)$.

The rest of the paper is organized as follows. In Section 2, we determine $\zeta_h(Q_n) = 2^h(n-h)$ for $h \leq n-2$ and $\eta_h(Q_n) = 2^h(n-h)$ for $h \leq n-1$. In Section 3, we determine $\zeta_h(S_n) = \eta_h(S_n) = h!(n-h)$ for $1 \leq h \leq n-1$. In Section 4, we determine $\eta_3(B_n) = 6(n-3)$ for $n \geq 4$ and point out a flaw in the proof of this conclusion in [13]. A conclusion is in Section 5.

For graph terminology and notation not defined here we follow Xu [10]. For a subset X of vertices in G, we do not distinguish X and the induced subgraph $G[X]$.

2. Hypercubes

The hypercube Q_n has the vertex-set consisting of 2^n binary strings of length n, two vertices being linked by an edge if and only if they differ in exactly one position. Hypercubes Q_1, Q_2, Q_3, Q_4 are shown in Fig. 1.