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A 13k-kernel for planar feedback vertex set via region decomposition ☆

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ABSTRACT

We show a kernel of at most 13k vertices for the Feedback Vertex Set problem restricted to planar graphs, i.e., a polynomial-time algorithm that transforms an input instance (G, k) to an equivalent instance with at most 13k vertices. To this end we introduce a few new reduction rules. However, our main contribution is an application of the region decomposition technique in the analysis of the kernel size. We show that our analysis is tight, up to a constant additive term.

1. Introduction

A *feedback vertex set* in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that $G - S$ is a forest. In the Feedback Vertex Set problem, given a graph G and integer k one has to decide whether G has a feedback vertex set of size k. This is one of the fundamental NP-complete problems, in particular it is among the 21 problems considered by Karp [12]. It has applications e.g. in operating systems (see [18]), VLSI design, synchronous systems and artificial intelligence (see [9]).

In this paper we study kernelization algorithms, i.e., polynomial-time algorithms which, for an input instance (G, k) either conclude that G has no feedback vertex set of size k or return an equivalent instance (G', k'), called *kernel*. In this paper, by the size of the kernel we mean the number of vertices of G'. Burrage et al. [6] showed that Feedback Vertex Set has a kernel of size $O(k^{11})$, which was next improved to $O(k^3)$ by Bodlaender [4] and to $4k^2$ by Thomassé [19]. Actually, as argued by Dell and van Melkebeek [8] the kernel of Thomassé can be easily tuned to have the number of *edges* bounded by $O(k^2)$. This cannot be improved to $O(k^{2-\epsilon})$ for any $\epsilon > 0$, unless coNP $\subseteq$ NP/poly [8].

In this paper we study Planar Feedback Vertex Set problem, i.e., Feedback Vertex Set restricted to planar graphs. Planar versions of NP-complete graph problems often enjoy kernels with $O(k)$ vertices. Since an n-vertex planar graph has $O(n)$ edges, this implies they have $O(k)$ edges, and hence are called *linear kernels*. The first nontrivial result of that kind was presented in the seminal work of Alber, Fellows and Niedermeier [2] who showed a kernel of size 335k for Planar Dominating Set. One of the key concepts of their paper was the region decomposition technique in the analysis of the

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kernel size. Roughly, in this method the reduced plane instance is decomposed into $O(k)$ *regions* (i.e. subsets of the plane) such that every region contains $O(1)$ vertices of the graph. It was next applied by Guo and Niedermeier to a few more graph problems [11]. In fact it turns out that for a number of problems on planar graphs, including Planar Dominating Set and Planar Feedback Vertex Set, one can get a kernel of size $O(k)$ by general method of protrusion decomposition [10]. However, in this general algorithm the constants hidden in the O notation are very large, and researchers keep working on problem-specific linear kernels with the constants as small as possible [7,17,20,16,15].

In the case of Planar Feedback Vertex Set, Bodlaender and Penninkx [3] gave an algorithm which outputs a kernel of size at most $112k$. This was next improved by Abu-Khzam and Khuzam [1] to $97k$. Very recently, and independently of our work, Xiao [21] has presented an improved kernel of $29k$ vertices. However, neither of these papers uses the region decomposition. Indeed, it seems non-obvious how the regions of the region decomposition can be defined for Planar Feedback Vertex Set. Instead, the authors of the previous works cleverly apply simple bounds on the number of edges in general and bipartite planar graphs. Moreover, for certain problems these methods turned out to give better results and simpler proofs than those based on region decomposition, see e.g., the work of Wang, Yang, Guo and Chen [20] on Connected Vertex Cover, Edge Dominating Set, and Triangle Packing in planar graphs improving previous results of Guo and Niedermeier [11].

Somewhat surprisingly, in this work we show that region decomposition can be successfully applied to Planar Feedback Vertex Set, and moreover it gives much tighter bounds than the previous methods. Furthermore, we add a few new reduction rules to improve the bound even further, to $13k$. More precisely, we show the following result.

Theorem 1. *There is an algorithm that, given an instance (G, k) of* Planar Feedback Vertex Set, *either reports that G has no feedback vertex set of size k or produces an equivalent instance with at most $13k - 24$ vertices. The algorithm runs in expected $O(n)$ time, where n is the number of vertices of G.*

We use the region decomposition approach in a slightly relaxed way: the regions are the faces of a k-vertex plane graph and the number of vertices of the reduced graph in each region is linear in the length of the corresponding face. We show that this gives a *tight bound*, i.e., we present a family of graphs which can be returned by our algorithm and have $13k - O(1)$ vertices.

A natural question is how far our result is from being optimal (assuming, say, $P \neq NP$). The only technique to lower bound the constant in the kernel size we are aware of is by studying the parametric dual, i.e., the same problem parameterized by $n - k$ (see [7]). The parametric dual of Planar Feedback Vertex Set is the Planar Maximum Induced Forest problem, where given a planar graph G and integer k we ask whether G contains an induced forest of at least k vertices. By the result of Borodin [5] every planar graph has a proper 5-coloring in which the union of any two color classes induces a forest. It follows that there an induced forest of size at least $\frac{2}{5}n$. Hence, Planar Maximum Induced Forest has a kernel of size at most $\frac{5}{2}k$, namely if $k \geq \frac{2}{5}n$ we return a trivial YES-instance, and otherwise we return the original graph. As discovered by Chen et al. [7], if the dual problem admits a kernel of size at most αk, then the original problem has no kernel of size at most $(\alpha/(\alpha - 1) - \epsilon)k$, for any $\epsilon > 0$, unless $P = NP$. It follows that Planar Feedback Vertex Set has no kernel of size at most $(\frac{5}{3} - \epsilon)k$, for any $\epsilon > 0$, unless $P = NP$. This is clearly quite far from our upper bound of $13k$. However, we suppose that the lower bound is not very tight.

Organization of the paper In Section 2 we present a kernelization algorithm which is obtained from the algorithms in [3,1] by generalizing a few reduction rules, and adding some completely new rules. In Section 3 we present an analysis of the size of the kernel obtained by our algorithm. In the analysis we assume that in the reduced graph, for every induced path with ℓ internal vertices, the internal vertices have at least three neighbors outside the path. Based on this, we get the bound of $(2\ell + 3)k - (4\ell + 4)$ for the number of vertices in the kernel. In Section 2 we present reduction rules which guarantee that in the kernel $\ell \leq 6$, resulting in the kernel size bound of $15k - 28$. To get the claimed bound of $13k - 24$ vertices in Section 4 we present a complex set of reduction rules, which allow us to conclude that $\ell \leq 5$. In Section 5 we discuss the running time of the algorithm. Finally, in Section 6 we discuss possibilities of further research.

Notation In this paper we deal with multigraphs, though for simplicity we refer to them as graphs. (Even if the input graph is simple, our algorithm may introduce multiple edges.) By the *degree* of a vertex x in a multigraph G, denoted by $\deg_G(x)$, we mean the number of edges incident to x in G. By $N_G(x)$, or shortly $N(x)$, we denote the set of neighbors of x, while $N[x] = N(x) \cup \{x\}$ is the closed neighborhood of x. Note that in a multigraph $|N_G(x)| \leq \deg_G(x)$, but the equality does not need to hold. The neighborhood of a set of vertices S is defined as $N(S) = (\bigcup_{v \in S} N(v)) \setminus S$, while the closed neighborhood of S is $N[S] = (\bigcup_{v \in S} N(v)) \cup S$. For a face f in a plane graph, a *facial walk* of f is the shortest closed walk induced by all edges on the boundary of f. The *length* of f, denoted by $d(f)$ is the length of its facial walk.

2. Our kernelization algorithm

In this section we describe our algorithm which outputs a kernel for Planar Feedback Vertex Set. The algorithm exhaustively applies reduction rules. Each reduction rule is a subroutine which finds in polynomial time a certain structure in the graph and replaces it by another structure, so that the resulting instance is equivalent to the original one. More precisely,