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Simultaneously moving cops and robbers

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A R T I C L E I N F O

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A B S T R A C T

In this paper we study the *concurrent cops and robber (CCCR)* game. CCCR follows the same rules as the classical, turn-based game, except for the fact that the players move *simultaneously*. The cops' goal is to capture the robber and the *concurrent cop number* of a graph is defined as the minimum number of cops which guarantees capture. For the variant in which it is required to capture the robber in the *shortest possible time*, we let time to capture be the *payoff function* of CCCR; the (game theoretic) *value* of CCCR is the optimal capture time and (cop and robber) *time optimal strategies* are the ones which achieve the value of the game. In this paper we prove the following.

1. For every graph G, the concurrent cop number is equal to the "classical" cop number.
2. For every graph G, CCCR has a value, the cops have an optimal strategy and, for every $\varepsilon > 0$, the robber has an ε-optimal strategy.

1. Introduction

In this paper we study the *concurrent cops and robber* game. In the classical CR game [15,16] each player observes the other player's move before he performs his own. On the other hand, in concurrent CR the players move *simultaneously*. In all other aspects, the concurrent game (henceforth CCCR) follows the same rules as the classical, turn-based game (henceforth TBCR).

The CCCR game (similarly to TBCR) can be considered as either a *game of kind* (the cops' goal is to capture the robber) or a *game of degree* (the cops' goal is to capture the robber *in the shortest possible time*).[1]

This paper is organized as follows. In Section 2 we define preliminary concepts and notation and use these to define the CCCR game rigorously. In Section 3 we concentrate on the "game of kind" aspect: we define the *concurrent cop number* $\widetilde{c}(G)$ and prove that, for every graph G, it is equal to the "classical" cop number $c(G)$. In Section 4 we concentrate on the "game of degree" aspect: we equip CCCR with a *payoff function* (namely the time required to capture the robber) and prove that (a) CCCR has a game theoretic *value*, (b) the cops have an optimal strategy and (c) for every $\varepsilon > 0$ the robber has an ε-optimal strategy; in addition we provide an algorithm for the computation of the value and the optimal strategies. In Section 5 we discuss related work. Finally, in Section 6 we present our conclusions and future research directions.

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[1] This terminology is due to Isaacs [8].

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2. Preliminaries

In this section, as well as in the rest of the paper, we will mainly concern ourselves with the case of a single cop; this is reflected in the following definitions and notation. In case $K > 1$ cops are considered, this will be stated explicitly; the extension of definitions and notation is straightforward.

2.1. Definition of the CCCR game

Both CCCR and TBCR are played on an undirected, simple and connected graph $G = (V, E)$ by two players called C and R. Player C, controlling K cops (with $K \geq 1$) pursues a single robber controlled by player R (we will sometimes call both the cops and robber *tokens*). We assume the reader is familiar with the rules of TBCR and proceed to present the rules of CCCR for the case of $K = 1$ (a single cop).

1. The game starts from given initial positions: the cop is located at $x_0 \in V$ and the robber at $y_0 \in V$.
2. At the t-th round ($t \in \mathbb{N}$) C moves the cop to $x_t \in N[x_{t-1}]$ and simultaneously R moves the robber to $y_t \in N[y_{t-1}]$.[2]
3. At every round both players know the current cop and robber location (and remember all past locations).
4. A *capture* occurs at the smallest $t \in \mathbb{N}$ for which either of the following conditions holds:
   (a) The cop is located at x_t, the robber is located at y_t, and $x_t = y_t$. This capture condition is the same as in TBCR.
   (b) The cop is located at x_{t-1} and moves to y_{t-1}, while the robber is located at y_{t-1} and moves to x_{t-1}. We will call this "*en passant*" capture; it does not have an analog in TBCR.
5. C wins if capture takes place for some $t \in \mathbb{N}$. Otherwise, R wins. The game analysis becomes easier if we assume that the game always lasts an infinite number of rounds; if a capture occurs at t_c, then we will have $x_t = y_t = x_{t_c}$ for all $t \geq t_c$.

We will denote the above defined game, played on graph $G = (V, E)$ and starting from initial position $(x, y) \in V^2$ by $\Gamma^G_{(x,y)}$. In case the game is played with K cops, it will be denoted by $\Gamma^{G,K}_{(x,y)}$ (in this case $x \in V^K$).

2.2. Nomenclature and notation

The following quantities will be used in the subsequent analysis (once again, we present definitions for the case of $K = 1$). Some of them require two separate definitions: one for TBCR and another for CCCR.

Definition 2.1. A *position* in TBCR is a triple (x, y, P) where $x \in V$ is the cop location, $y \in V$ is the robber location and $P \in \{C, R\}$ is the player whose turn it is to move. We also have $|V| + 1$ additional positions:

1. the position $(\emptyset, \emptyset, C)$ corresponds to the beginning of the game, before either player has placed his token;
2. the positions $(x, \emptyset, R)$, $x \in V$, correspond to the phase of the game in which C has placed the cop but R has not placed the robber.

The set of all TBCR positions is denoted by $S = V \times V \times \{C, R\}$.

Definition 2.2. A *position* in CCCR is a pair $(\widetilde{x}, \widetilde{y})$ where $\widetilde{x} \in V$ is the cop location and $\widetilde{y} \in V$ is the robber location. The set of all CCCR positions is denoted by $\widetilde{S} = V \times V$.

Definition 2.3. A *history* is a position sequence of finite or infinite length. The set of all game histories of *any* finite length is denoted by S^* for TBCR and $\widetilde{S}^*$ for CCCR. The set of all infinite game histories is denoted by S^∞ for TBCR and $\widetilde{S}^\infty$ for CCCR.

In both TBCR and CCCR, the players' *moves* are graph nodes, e.g., $x, y \in V$. Given the current position and the next move (in TBCR) or moves (in CCCR), the next game position is determined by the *transition function*, which encodes the rules of the respective game.

Definition 2.4. In TBCR, the *transition function* $Q : S \times V \to S$ is defined as follows:

$$\text{when } x = y: \qquad Q\left((x, y, C), x'\right) = (x, x, R)$$
$$\text{when } x \neq y \text{ and } x' \in N[x]: \qquad Q\left((x, y, C), x'\right) = \left(x', y, R\right)$$

[2] $N[u]$ denotes the *closed neighborhood* of node u, i.e., the set containing u itself and all nodes connected to u by an edge.