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The extra connectivity of bubble-sort star graphs ☆

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ABSTRACT

Connectivity plays an important role in measuring the fault tolerance of a multiprocessor system in the case of vertices failures. Extra connectivity is an important indicator of a network's ability for diagnosis and fault tolerance. In this paper, we analyze the fault tolerance ability of the bubble-sort star network, denoted by BS_n, a well-known interconnection network proposed for multiprocessor systems, and establish the g-extra connectivity for $1 \le g \le 3$.

1. Introduction

With the rapid development of technology, a multiprocessor system may contain thousands or even more processors co-operating to solve large-scale computing problems. As a significant increase in the number of processors, the processor failure is inevitable. Hence, fault diagnosis and fault tolerance of interconnection networks have become increasingly important.

The interconnection network is often modeled as an undirected graph, where vertex and edge correspond the processor and the link between two processors, respectively. In a graph G, the connectivity of G, denoted by $\kappa(G)$, is the number, such that at least $\kappa(G)$ vertices have to be removed to disconnect the graph G. The connectivity of a graph is closely related to its reliability and fault tolerability. However, it is no more than the minimum vertex degree of the graph. Hence the connectivity of a graph may not be the best indicator of its reliability and fault tolerability. Several variants of connectivity were proposed to establish a better relationship between connectivity and fault tolerability. Among the proposed connectivity, the *g-extra connectivity* in [5] is widely used. For example, see [2,3,6,8–15].

Clearly, the star graph owns many attractive properties except the embeddability as well as the Bubble-sort graph is simple and possesses some desirable features except the long diameter. So we may expect that the Bubble-sort star graph will combine the advantages of both graphs (see [1,4,7]). The remainder of this paper is organized as follows: Section 2 introduces some necessary definitions and notations. Section 3 states and proves some combinatorial and fault-tolerant properties of Bubble-sort star graph BS_n. Section 4 explores the g-extra connectivities of BS_n, where $1 \le g \le 3$. Section 5 concludes this paper.

2. Preliminaries

The topology of an interconnection network is often modeled as an undirected graph, where vertex and edge represent the processor and the link between two processors, respectively. Throughout this paper, we only consider simple, undirected

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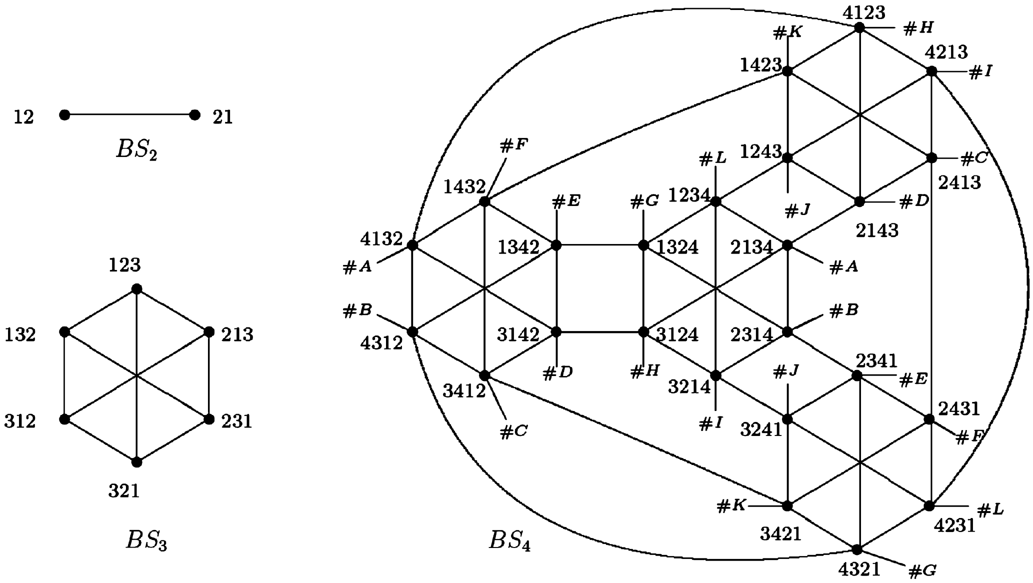

Fig. 1. The Bubble-sort star graphs BS_2, BS_3 and BS_4.

and connected graphs. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. For $v_i \in V(G)$, the degree of v_i, written by $d_G(v_i)$, is the number of edges incident with v_i. Let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of G, respectively. If $\Delta(G) = \delta(G)$, then the graph is *regular*. The set of neighbors of a vertex v_i in G is denoted by $N_G(v_i)$. Let $cn(G)$ be the maximum number of common neighbors between any two vertices in G.

For a subset S of $V(G)$, $G[S]$ is a subgraph of G induced by S, which has vertex set S and edge set $\{uv | uv \in E(G), u, v \in S\}$. The *neighborhood set* of S in G is defined as $N_G(S) = (\cup_{u \in S} N_G(u)) - S$. We will use $G - S$ to denote the subgraph $G[V(G) - S]$. The minimum size of a vertex set $S \subseteq V(G)$ that the graph $G - S$ is disconnected or has only one vertex, denoted by $\kappa(G)$, is the *connectivity* of G. A *path* in a graph is a sequence of distinct vertices so that there is an edge joining consecutive vertices. A *cycle* on three or more vertices is a path where there is an edge joining the first and the last vertices. We will use P_k or C_k to denote a path or a cycle of order k, respectively.

Now we give the definitions of the g-extra connectivity and the bubble-sort star graph BS_n.

Definition 2.1. [5] Given a graph G and a nonnegative integer g, the *g-extra connectivity* of G, denoted by $\kappa_0^{(g)}(G)$ is the minimum cardinality of a vertex cut S of G, if it exists, whose deletion leaves each remaining component of $G - S$ with more than g vertices.

Let $[a, b] = \{x : x$ is an integer with $a \le x \le b\}$, where a and b are integers. We denote "$\circ$" to be an operation such that $u = v \circ (i, j)$, for any $u = x_1 x_2 \cdots x_i \cdots x_j \cdots x_n$, $v = x_1 x_2 \cdots x_j \cdots x_i \cdots x_n$, where $x_i \in [1, n]$ and $x_i \ne x_j (i \ne j,$ and $i, j \in [1, n])$.
Now we give the definition of bubble-sort star graph.

Definition 2.2. [4] The bubble-sort star graph BS_n has $n!$ vertices, each of which has the form $u = x_1 x_2 \cdots x_n$, where $x_i \in [1, n]$ and $x_i \ne x_j$ for $i \ne j$, where $i, j \in [1, n]$. Any two vertices u and v of $V(BS_n)$ are adjacent if and only if $v = u \circ (1, i)$ for $i \in [2, n]$, or $v = u \circ (i - 1, i)$ for $i \in [3, n]$.

Clearly, BS_n is $(2n - 3)$-regular and vertex symmetry. Moreover, it is bipartite and Hamiltonian. BS_n can be partitioned into n subgraphs $BS_n^1, BS_n^2, \cdots, BS_n^n$, where every vertex $u = x_1 x_2 \cdots x_n \in V(BS_n^i)$ has a fixed integer i in the last position x_n for $i \in [1, n]$. For any vertex $u \in V(BS_n^i)$, we denote $u^+ = u \circ (1, n)$, $u^- = u \circ (n - 1, n)$, $N_u^+ = \{u^+, u^-\}$. Let $E_{i,j}(BS_n) = E_{BS_n}(V(BS_n^i), V(BS_n^j))$ for simplicity, where $E_{BS_n}(V(BS_n^i), V(BS_n^j))$ denotes the edge set of BS_n with one end in $V(BS_n^i)$ and other end in $V(BS_n^j)$. It is obvious that BS_n^i is isomorphic to BS_{n-1} for $i \in [1, n]$. Fig. 1 illustrates BS_2, BS_3 and BS_4, respectively.

3. Fault tolerance properties of BS_n

In this section, we establish the sufficient conditions to determine the lower bound of extra connectivity of BS_n. Throughout this Section, we will use F to denote a subset of $V(BS_n)$. Denote $F_i = F \cap V(BS_n^i)$ for $i \in [1, n]$ and assume that $|F_1| \ge |F_2| \ge \cdots \ge |F_n|$. For any $I \subseteq [1, n]$, we use BS_n^I to denote the subgraphs of BS_n induced by $\cup_{i \in I} V(BS_n^i)$ and $F^I = \cup_{i \in I} V(F_i)$ is a subset of $V(BS_n)$. Note that BS_n is a bipartite graph which implies BS_n contains no odd cycle.