Theoretical Computer Science 645 (2016) 128–146

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Non-local estimators: A new class of multigrid convergent length estimators ☆

Loïc Mazo *, Étienne Baudrier

ICube, University of Strasbourg, CNRS, 300 Bd Sébastien Brant, CS 10413, 67412 Illkirch, France

A R T I C L E   I N F O

Article history:
Received 25 August 2015
Received in revised form 23 April 2016
Accepted 6 July 2016
Available online 16 July 2016
Communicated by R. Gavaldà

Keywords:
Discrete geometry
Length estimation
Multigrid convergence

A B S T R A C T

An interesting property for curve length digital estimators is the convergence toward the continuous length and the associate convergence speed when the grid spacing tends to zero. On the one hand, DSS based estimators were proved to converge but only under some convexity and smoothness or polygonal assumptions. On the other hand, the sparse estimators were introduced in a previous paper by the authors and their convergence for Lipschitz functions was proved without convexity assumption. Here, a wider class of estimators, the *non-local estimators*, is defined that intends to gather sparse estimators and DSS based estimators. Their convergence is proved and an error upper bound for a large class of functions is given.

1. Introduction

This paper focuses on a classical digital problem: the length estimation. The problem is to estimate the length of a continuous curve S knowing a digitization of S. As information is lost during the digitization step, there is no reliable estimation without *a priori* knowledge. From a theoretical point of view, a classical criterion to evaluate the quality of a geometric feature estimator is the possession, or not, of the *(multigrid) convergence* property, that is the estimation convergence toward the continuous curve feature when the grid spacing tends to zero. The local estimators based on a segmentation of the digital curve in patterns whose size is a constant that does not depends upon the grid spacing do not satisfy the convergence property even for straight line segments [1]. The adaptive estimators based on a segmentation in Maximal Digital Straight Segments (MDSS) or based on a Minimum Length Polygon (MLP) satisfy the convergence property for smooth, or polygonal, closed simple curves under assumption of convexity [2]. The semi-local estimators [3], and the sparse estimators [4], both based on a segmentation of the curve in patterns whose size only depends upon the grid spacing, verifies the convergence property[1] without convexity hypothesis, for smooth functional curves of class C^2 with the former and for Lipschitz curves with the latter. A new class of length estimators is presented here, the *non-local estimators*, that aims to encompass the sparse estimators and the MDSS based estimators.

The paper is organized as follows. In Section 2, some necessary notations and conventions are recalled, then the existing estimators and their convergence properties are detailed. In Section 3, the non-local estimators are defined and the multigrid convergence property is proved for Lipschitz functions under some assumptions satisfied by sparse estimators and MDSS

☆ This work was supported by the Agence Nationale de la Recherche through contract ANR-2010-BLAN-0205-01.
* Corresponding author.
E-mail address: loic.mazo@unistra.fr (L. Mazo).

[1] Actually, the convergence of a semi-local estimator depends upon some choice made in its definition (see Section 2.4).

http://dx.doi.org/10.1016/j.tcs.2016.07.007
0304-3975/

based estimators. Furthermore, an upper bound on the error of the estimator is exhibited for a wide subclass of the Lipschitz functions. Section 4 provides some illustrations about the convergence speed and a comparison of the estimations for different kind of non-local estimators. Section 5 concludes the article and gives directions for future works.

2. Background

2.1. Discretization models

In this work, we have restricted ourselves to the digitization of function graphs. So, let us consider a continuous function $g : [a,b] \to \mathbb{R}$ $(a < b)$, its graph $\mathcal{C}(g) = \{(x, g(x)) \mid x \in [a,b]\}$ and a positive real number h, the *grid spacing*. An orthogonal grid in the Euclidean space $\mathbb{R}^2$ is assumed, whose set of grid points is $h\mathbb{Z}^2$.

The common methods to model the digitization of the graph $\mathcal{C}(g)$ with a grid spacing h are closely related to each others. In this paper, an *object boundary quantization* (OBQ) is assumed. This method associates to the graph $\mathcal{C}(g)$ the *h-digitization* set

$$\mathcal{D}(g,h) = \{(kh, \left\lfloor \frac{g(kh)}{h} \right\rfloor h) \mid k \in \mathbb{Z} \text{ and } kh \in [a,b]\},$$

where $\lfloor\cdot\rfloor$ denotes the floor function. The set $\mathcal{D}(g,h)$ contains the uppermost grid points which lie in the hypograph of g, hence it can be understood as a part of the boundary of a solid object. Provided the slope of g is limited by 1 in modulus, $\mathcal{D}(g,h)$ is an 8-connected digital curve. Observe that if g is a function of class C^1 such that the set $\{x \in [a,b] \mid |g'(x)| = 1\}$ is finite, then by symmetries on the graph $\mathcal{C}(g)$, it is possible to come down to the case where $\left|g'\right| \le 1$. Nevertheless, in this article, no assumption is made on the slope of the function g and by *discrete curve* we mean the graph of a function $\gamma : I \to \mathbb{Z}$ where I is an interval of $\mathbb{Z}$.

In the sequel of the article, for any function $f : [a,b] \to \mathbb{R}$, $L(f)$ denotes the length of the graph $\mathcal{C}(f)$ according to Jordan's definition of length:

$$L(f) = \sup_{a=x_0<x_1<\dots<x_n=b} \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2},$$

where the supremum is taken over all possible partitions of $[a,b]$ and n is unbounded.

2.2. Local estimators

Local length estimators (see [5] for a short review) are based on parallel computations of the length of fixed size segments of a digital curve. For instance, an 8-connected curve can be split into 1-step segments. For each segment, the computation returns 1 whenever the segment is parallel to the axes and $\sqrt{2}$ when the segment is diagonal. Then all the results are added to give the curve length estimation.

This kind of local computation is the oldest way to estimate the length of a curve and has been widely used in image analysis. Nevertheless, it has not the convergence property. In [1], Kulkarni et al. introduce a general definition of local length estimation with sliding segments and prove that such computations cannot give a convergent estimator for straight lines whose slope is small (less than the inverse of the size of the sliding segment). In [6], a similar definition of local length estimation is given with disjoint segments. Again, it is shown that the estimator failed to converge for straight lines (with irrational slopes). This behavior is experimentally confirmed in [2] on a test set of five closed curves. Moreover, the non-convergence is established in [7,8] for almost all parabolas.

2.3. Adaptive estimators: DSS and MLP

Adaptive length estimators gather estimators relying on a segmentation of the discrete curve that depends on each point of the curve: a move on a point can change the whole segmentation. Unlike local estimators, it is possible to prove the convergence property of adaptive length estimators under some assumptions. Adaptive length estimators include two families of length estimators, namely the Maximal Digital Straight Segment (MDSS) based length estimators and the Minimal Length Polygon (MLP) based length estimators.

Definition and properties of MDSS can be found in [2,9,10]. Efficient algorithms have been developed for segmenting curves or function graphs into MDSS and to compute their characteristics in a linear time [9–11]. The decomposition in MDSS is not unique and depends on the start-point of the segmentation and on the curve travel direction. The convergence property of MDSS estimators was proved for convex polygons whose MDSS polygonal approximation[2] is also convex [12, Th. 13 and the proof][3]: given a convex polygon $\mathcal{C}$ and a grid spacing h (below some threshold), the error between the estimated length $L_{\text{est}}(\mathcal{C}, h)$ and the true length of the polygon $L(\mathcal{C})$ is such that

[2] Though the digitization of a convex set is digitally convex, it does not mean that a polygonal curve related to a convex polygonal curve via a MDSS segmentation process is also convex.

[3] The hypothesis on the convexity of the MDSS polygon is not assumed in the statement of the theorem but it appears in the proof.