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Straight skeletons with additive and multiplicative weights and their application to the algorithmic generation of roofs and terrains ☆

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ABSTRACT

We introduce additively-weighted straight skeletons as a new generalization of straight skeletons. An additively-weighted straight skeleton is the result of a wavefront-propagation process where, unlike in previous variants of straight skeletons, wavefront edges do not necessarily begin to move at the start of the propagation process but at later points in time. We analyze the properties of additively-weighted straight skeletons and show how to compute straight skeletons with both additive and multiplicative weights, i.e., where input edges are allowed to move at different speeds and may start at different times.

We then show how to use additively-weighted and multiplicatively-weighted straight skeletons to generate roofs and terrains for polygonal shapes such as the footprints of buildings or river networks. As a result, we are able to automatically generate roofs and terrains where the individual facets have different inclinations and may start at different heights.

1. Introduction

1.1. Motivation and prior work

Straight skeletons were introduced to computational geometry over 20 years ago by Aichholzer et al. [1]. Suppose that the edges of a simple polygon P move inwards with unit speed in a self-parallel manner, thus generating mitered offsets inside of P. Then the (unweighted) straight skeleton of P is the geometric graph whose edges are given by the traces of the vertices of the shrinking mitered offset curves of P; see Fig. 1 and Section 2.

Multiplicatively-weighted straight skeletons were first mentioned by Aichholzer and Aurenhammer [2] and then by Eppstein and Erickson [3]. Roughly, the presence of multiplicative weights implies that the edges of P are allowed to move inwards at different speeds. Recently, multiplicatively-weighted straight skeletons were studied in detail by Biedl et al. [4], who analyzed under which conditions properties of the unweighted skeleton carry over to the weighted pendant.

Unweighted and multiplicatively-weighted straight skeletons are known to have applications in diverse fields. Aurenhammer [5] investigates fixed-share decompositions of convex polygons using skeletons with specific positive multiplicative weights. Barequet et al. [6] employ multiplicatively-weighted straight skeletons as a theoretical tool for computing (unweighted) straight skeletons in three-space. Barequet and Yakersberg [7] morph shapes by means of their straight skeletons. Tomoeda and Sugihara use straight skeletons to create signs with an illusion of depth [8], and Sugihara also uses multiplicatively-weighted skeletons in the design of pop-up cards [9]. Haunert and Sester [10] apply them for topology-preserving area collapsing in geographic information systems (GIS). In another GIS application, Vanegas et al. [11] use straight skeletons for generating parcels in urban modeling.

The automatic generation of roofs of buildings based on straight skeletons of their footprints (i.e., bird's eye view) has also received wide-spread attention in large-scale urban modeling. E.g., Larive and Gaildrat [12], Müller et al. [13], and Buron et al. [14] combine GIS data and shape grammars with production rules to generate roofs for buildings. As a starting point or if a purely grammar-based generation is not possible, they resort to roofs obtained from straight skeletons. The roofs in the recent work by Sugihara [15,16] are based on straight skeletons as well. Furthermore, Laycock and Day [17] and Kelly and Wonka [18] use multiplicatively-weighted straight skeletons for modeling roofs in more realistic ways. Roofs created by straight skeletons are limited to hip roofs and, with some postprocessing, gable roofs. Their ridges tend to be parallel to long edges of the footprint of the building. Typically, such roofs will not have ridges that are perpendicular to long (parallel) edges of the footprint.

A problem closely related to the generation of roofs is the (re-)construction of terrains. For instance, we might be given a river map together with estimates of the slopes of the terrain.

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Straight skeletons offer a promising approach to both roof generation and terrain construction. Of course, straight skeletons are not the only means for generating terrains; see, for instance, [19–21].

1.2. Our contribution

We introduce an additively-weighted straight skeleton as a new generalization of straight skeletons: If additive weights are present, then edges of the input need not all start to move at the same time. We analyze the properties of additively-weighted straight skeletons and show how to extend the standard algorithmic framework for computing straight skeletons (based on wavefront propagation) to additively-weighted straight skeletons.

We also argue that this framework allows to handle both additive and multiplicative weights. Multiplicative weights translate to different speed functions for the input edges, but each speed stays constant throughout the entire movement of the edge. As a matter of fact, in our framework any speed function that remains piecewise constant could be used for an edge, thus extending traditional straight skeletons even further. In the limit, at the cost of combinatorial complexity, piecewise constant speed functions support arbitrary edge velocity profiles.

The input for our algorithm need not be constrained to simple polygons. Rather, any planar straight-line graph (PSLG), i.e., any collection of straight-line segments that do not intersect pairwise except at common end-points, forms a permissible input.

Combining both additive and multiplicative weights yields input edges that (1) are allowed to move at different speeds and (2) may start at different times. As a result, we get a process for automatic generation of roofs or terrains where the individual facets have different inclinations and may start at different heights. In particular, additive weights allow for gable roofs without post-processing, with the ridge being perpendicular to some long edge of the footprint. General piecewise constant speed functions result in piecewise linear surfaces (roof, terrain, etc.) where individual facets may have kinks.

As for unweighted straight skeletons, additively- and multiplicatively-weighted straight skeletons come with an important property: A raindrop that hits a facet of a surface generated by means of a weighted straight skeleton is guaranteed to run off. That is, no local minima can occur on the surface.

2. Preliminaries

Wavefront Propagation Process. Let P denote a simple polygon. The straight skeleton of P is defined by means of a wavefront propagation process. The wavefront $\mathcal{W}_P(t)$ is a set of wavefront polygons and changes with time t. Initially, at time zero, $\mathcal{W}_P(0)$ consists only of P. Then, as time increases, the edges of $\mathcal{W}_P(t)$ move towards the interior of P at unit speed in a self-parallel manner, thereby preserving incidences. Thus, the vertices of $\mathcal{W}_P(t)$ move along the angular bisectors of polygon edges, and the wavefront corresponds to a mitered offset of P; see Fig. 1.

To maintain the planarity of the wavefront during the propagation process, Aichholzer et al. [1] resolve non-planarities when they occur:

- In an *edge event*, an edge of the wavefront has shrunk to zero length. This edge is removed from the wavefront, resulting in the two adjacent edges becoming neighbors.
- In a *split event*, a reflex vertex v reaches another part of the wavefront. (A vertex v of P is called reflex if the interior angle at v is greater than 180°, and convex if it is less than 180°; tangential vertices with interior angle equal to 180° can be ignored during the wavefront propagation.) The wavefront is split at this locus, and two separate polygons replace the previous polygon to restore planarity of the wavefront after the event. Typically this will happen when v reaches the interior of a wavefront edge. However, if v reaches another vertex then more complex interactions are possible, resulting in non-elementary events [22].

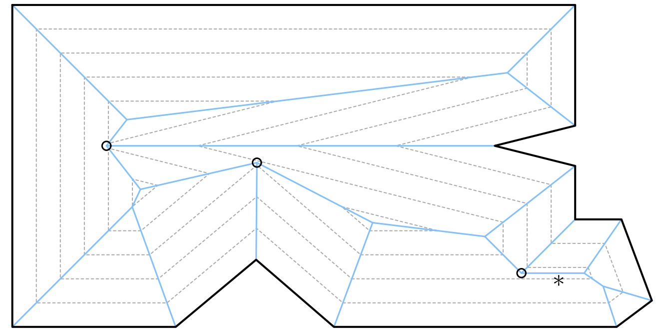

Fig. 1. Polygon (bold) with its straight skeleton. A family of mitered offset curves, i.e., the wavefronts at different times, is shown in dotted gray. The straight skeleton nodes marked with ∘ are the result of split events; the others come from edge events. The straight skeleton arc marked with ∗ is one that was added when two parallel wavefront edges moved into each other.

Since the wavefront moves inwards within a polygon of finite extension, at some point $\bar{t}$ in time all wavefront polygons will have collapsed, thus resulting in $\mathcal{W}_P(\bar{t})$ being the empty set. At this time $\bar{t}$ the propagation process ends.

Straight Skeleton. The *straight skeleton* $\mathcal{S}(P)$ is the geometric graph whose edges are the traces of all vertices of $\mathcal{W}_P(t)$ over the entire propagation period. In addition, if two parallel wavefront edges move into each other during the wavefront propagation, then also the portion common to them is added to the straight skeleton while the portions that belong to only one of them remain in the wavefront [4]. The vertices of $S(P)$ are the endpoints of its edges. Fig. 1 shows wavefront polygons at different times and the resulting straight skeleton.

To avoid ambiguities, one generally refers to the edges of the straight skeleton as *arcs* and reserves the term *edges* for the input polygon and the wavefront. Likewise, the vertices of a straight skeleton are called *nodes*.

The straight skeleton of a polygon is a tree and each interior node of $\mathcal{S}(P)$ is of degree three for input in general position such that only elementary edge and split events occur during the wavefront propagation [1]. Since the vertices of the wavefront move along angular bisectors of edges of P, all arcs of the straight skeleton are straight-line segments.

Faces. The wavefront fragments of the polygon edge e at time t are contained in $\bar{e}+t\cdot n_e$, where $\bar{e}$ is the supporting line of e and n_e is its inward facing unit normal. We denote by $e(t)$ the (possibly empty) set of these wavefront fragments of edge e at time t. Every *face* of the straight skeleton is traced out by the fragments of exactly one input edge over time, i.e., $f(e) := \bigcup_{t\geq 0} e(t)$ for the face $f(e)$ of edge e. Furthermore, it is known that $f(e)$ is monotone with respect to e [1].

Roof Model. The roof model [1] raises the wavefront propagation into three-space, with the third (z-)coordinate being the time t. With P embedded in the xy-plane $t = 0$, the propagation of the wavefronts over time forms a polytope over P. This piecewise linear and continuous polytope $R(P) := \bigcup_{t\geq 0}(\mathcal{W}_P(t)\times\{t\})$ is called the *roof* of P. This roof is a terrain, i.e., it is a z-monotone surface where each line parallel to the z-axis intersects it at most once.

The roof model is a useful theoretical tool when dealing with straight skeletons as it makes some proofs easier. It is also directly