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Rectifying control polygon for planar Pythagorean hodograph curves ☆

Soo Hyun Kim, Hwan Pyo Moon *

Department of Mathematics, Dongguk University-Seoul, Seoul, 04620, Republic of Korea

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ABSTRACT

A Bézier control polygon is not appropriate to control a Pythagorean hodograph curve since it has redundant degrees of freedom. So we propose an alternative, which is the rectifying control polygon. A rectifying control polygon of a PH curve has the same degrees of freedom as the PH curve. It interpolates the end points of the PH curve, but not the end tangents. Most of all, it has the same arc length as the PH curve. In this paper, we present the method to compute the rectifying control polygon from the Bézier control polygon of the PH curve. We also present the procedure to compute the PH curves from a given rectifying control polygon. For the development of these algorithms, we employ the Gauss–Legendre quadrature method and the Bernstein–Vandermonde linear system.

1. Introduction

Pythagorean hodograph (PH) curves, introduced by Farouki and Sakkalis (1990), are a special type of polynomial curves, which have the unique property that their parametric speed functions are also polynomials of the curve parameter. The PH property enables us to compute the arc length of the curve exactly without numerical integration. The curve construction problem with prescribed arc length has been studied by Roulier (1993), Roulier and Piper (1996) using Bézier curves or even general parametric curves. Their methods rely on iterative numerical computations. Similar problems using PH curves were solved by Farouki (2016), Huard et al. (2014) in much simpler and elegant ways because PH curves have explicit expression of the arc length. Another important advantage of PH curves is that their offset curves are rational curves. So we do not need to rely on approximation algorithms for offset computation. One may consult Farouki (2007) for more details on PH curves from algebraic frameworks to practical applications.

Since PH curves are polynomial curves, they can be expressed as Bézier curves. However, not all Bézier curves are PH curves. Recently, Farouki et al. (2015) suggest the method to determine whether a given Bézier curve is in fact a PH curve, and to compute the parameters of the PH curve. They also presented the method of local modification of quintic PH spline curve while maintaining the PH property (Farouki et al., 2016). Since the Bézier control points of a PH curve should satisfy certain algebraic constraints, we cannot move a Bézier control point arbitrarily while preserving the PH condition. Actually a PH curve has fewer degrees of freedom than a Bézier curve of the same degree has. This situation suggests that a Bézier control polygon is not appropriate to "control" a PH curve. So we propose an alternative method to control PH curves, which is the *rectifying control polygon*.

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* Corresponding author.
E-mail addresses: sookim@dongguk.edu (S.H. Kim), hpmoon@dongguk.edu (H.P. Moon).

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A rectifying control polygon of a PH curve will be defined as a polygon which satisfies the properties: (1) it has the same degrees of freedom as the PH curve has, (2) it interpolates both end points of the PH curve, and (3) the sum of the lengths of polygon segments is the same as the length of the PH curve. The third condition above is the rectifying property of this polygon.

To achieve the rectifying property, we will utilize the Gauss–Legendre quadrature for the computation of the arc length of a PH curve. The Gauss–Legendre quadrature with n nodes produces the exact integral of a polynomial of degree $2n-1$. We can construct a rectifying control polygon of a PH curve by attaching the tangent vectors of the PH curve, evaluated at the Gauss–Legendre nodes and scaled by the Gauss–Legendre weights. One may find the details of the Gauss–Legendre quadrature in basic texts of numerical analysis such as Atkinson (1989), Atkinson and Han (2004).

The rest of the paper is structured as follows. In Section 2, we summarize the fundamental properties of PH curves and the Gauss–Legendre quadrature, then fix the notations. Section 3 is devoted to the construction of a rectifying polygon from a given PH curve. We here prove that the vertices of the rectifying polygon can be obtained by the convex combinations of the Bézier control points of the PH curve. In Section 4, we solve the inverse problems of Section 3. We construct PH curves from a given rectifying control polygon. In fact, there are multiple instances of PH curves for a given rectifying control polygon. We describe the procedure to compute these PH curves. This procedure can be formulated as a linear system whose coefficient is the Bernstein–Vandermonde matrix evaluated at the Gauss–Legendre nodes. Finally we conclude the paper in Section 5.

2. Preliminaries

When we deal with planar geometry, it is convenient to employ the complex number field. A point $\mathbf{z} = (x, y)$ in $\mathbb{R}^2$ is identified with the complex number $\mathbf{z} = x + \mathrm{i}\, y$ in $\mathbb{C}$. Similarly, a planar parametric curve $\mathbf{p}(t) = (x(t), y(t))$ can be identified with a complex valued function $\mathbf{p}(t) = x(t) + \mathrm{i}\, y(t)$. A Bézier curve $\mathbf{p}$ of degree n with complex Bézier control points $\mathbf{b}_k = x_k + \mathrm{i}\, y_k$ for $k = 0, \cdots, n$, can then be expressed as

$$\mathbf{p}(t) = \sum_{k=0}^{n} B_k^n(t)\mathbf{b}_k, \qquad t \in [0,1] \tag{1}$$

where $B_k^n(t)$ is the Bernstein polynomial defined by

$$B_k^n(t) = \binom{n}{k}(1-t)^{n-k}t^k.$$

Let $\triangle\mathbf{b}_k$ denote the forward difference of the k-th control point, i.e., $\triangle\mathbf{b}_k = \mathbf{b}_{k+1} - \mathbf{b}_k$. Then the derivative of the curve $\mathbf{p}$ is given by

$$\mathbf{p}'(t) = n\sum_{k=0}^{n-1} B_k^{n-1}(t)\triangle\mathbf{b}_k.$$

A planar polynomial curve $\mathbf{p}(t) = x(t) + \mathrm{i}\, y(t)$ is called a *Pythagorean hodograph (PH)* curve (Farouki and Sakkalis, 1990) if and only if its derivative $\mathbf{p}'(t) = x'(t) + \mathrm{i}\, y'(t)$ satisfies the Pythagorean condition

$$x'(t)^2 + y'(t)^2 = \sigma(t)^2$$

for some polynomial $\sigma(t)$. The algebraic structures of Pythagorean hodograph curves have been studied in many different contexts. We here refer to a few important results.

Theorem 1. *A planar polynomial curve $\mathbf{p}(t) = x(t) + \mathrm{i}\, y(t)$ of odd degree is a Pythagorean hodograph curve if and only if there exist polynomials $u(t)$ and $v(t)$ which satisfies*

$$x'(t) = u(t)^2 - v(t)^2,$$
$$y'(t) = 2u(t)v(t).$$

Theorem 2. *A planar polynomial curve $\mathbf{p}(t) = x(t) + \mathrm{i}\, y(t)$ of odd degree is a Pythagorean hodograph curve if and only if there exists a complex valued polynomial $\mathbf{z}(t)$ such that*

$$\mathbf{p}'(t) = \mathbf{z}(t)^2.$$

One may consult Farouki (1994) for the proof of the above theorems.

For a regular PH curve $\mathbf{p}(t)$ with $\mathbf{p}'(t) \neq \mathbf{0}$, the polynomial $\sigma(t)$ can be chosen as a positive polynomial which is the speed function of $\mathbf{p}(t)$, i.e.,