ARTICLE IN PRESS
COMAID:1593

Computer Aided Geometric Design ••• (••••) •••–•••

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Algebraic and algorithmic aspects of radical parametrizations ☆

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ARTICLE INFO

Article history:
Available online xxxx

Keywords:
Radical parametrization
Tracing index
Reparametrization
Implicitization

ABSTRACT

In this article algebraic constructions are introduced in order to study the variety defined by a radical parametrization (a tuple of functions involving complex numbers, n variables, the four field operations and radical extractions). We provide algorithms to implicitize radical parametrizations and to check whether a radical parametrization can be reparametrized into a rational parametrization.

1. Introduction

It is well known that in many applications dealing with geometric objects, parametric representations are very useful (see Hoschek and Lasser, 1993). In general, when one works with parametric representations of algebraic curves and surfaces, the involved functions are rational. Nevertheless it is well known that only genus 0 curves, and arithmetic and plurigenus 0 surfaces, have this property. Furthermore, even if we limit our parametrizations to the rational case, many of the geometric constructions in CAGD, as e.g. offsetting, conchoidal or cissoid constructions, do not propagate the rationality of the geometric object. This means that even though the original object is rational the new object is not in general.

One possible way to overcome this limitation is to work with piecewise approximate parametrizations. A second way is to extend the family of functions used in the parametric representation; for instance using radicals of polynomials. The latter is the frame of this paper. When dealing with curves, algorithms to parametrize with radicals can be found in Sendra and Sevilla (2011) and Harrison (2013) that cover the cases of genus less or equal to 6. In addition, in Sendra and Sevilla (2013) one can find algorithms to parametrize by radicals certain classes of surfaces. An additional interesting property of radical parametrizations is that the radical nature of a variety is preserved under geometric constructions of degree up to 4 (see Section 5 in Sendra and Sevilla, 2013 for further details). So, in particular, offsets and conchoids of radical varieties are radical (see Corollaries 5.2 and 5.3. in Sendra and Sevilla, 2013).

In this article we continue the exploration of radical parametrizations initiated in Sendra and Sevilla (2011) and Sendra and Sevilla (2013). Here we introduce a framework to manipulate these parametrizations in a rational way by means of rational auxiliary varieties and maps. This allows us to apply results of algebraic geometry to derive conclusions on the radical parametrization and its image. In short, to translate radical statements into rational ones. More precisely, from the

☆ Partially supported by the Spanish *Ministerio de Economía y Competitividad* under Project MTM2014-54141-P, and by *Junta de Extremadura* and FEDER funds (group FQM024).

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http://dx.doi.org/10.1016/j.cagd.2017.01.002
0167-8396/

theoretical point of view, we introduce the notion of radical variety associated to a radical parametrization, and we prove that it is irreducible and of dimension equal to the number of parameters in the parametrization. Furthermore, we introduce the notion of tracing index of radical parametrization that extends the notion of properness of rational parametrization (see Sendra and Winkler, 2001). In addition, we define an algebraic variety, that we call tower variety, which is birationally equivalent to the radical variety when the tracing index is 1. The most interesting property of the tower variety is that it encodes rationally the information of the radical parametrization. From the algorithmic point of view, we show how to compute generators of the radical variety and of the tower variety, in particular an implicitization algorithm for radical parametrizations, and how to compute the tracing index. As a potential application we present an algorithm to decide, and actually compute, whether a given radical parametrization can be reparametrized into a rational parametrization. Also, we show how the tower variety may help to compute symbolically integrals whose integrand is a rational function of radicals of polynomials.

The paper is structured as follows: in Section 2 we recall the notion of radical parametrization and we discuss how to represent them. In Section 3 we introduce the concept of radical variety and we prove some of the main properties. In Section 4 the tower variety is defined, properties are presented, and its application to check the reparametrizability of radical parametrizations into rational parametrizations is illustrated.

2. The notion of radical parametrization

A radical parametrization is, intuitively speaking, a tuple $\overline{x} = (x_1(\overline{t}), \ldots, x_r(\overline{t}))$ of functions of variables $\overline{t} = (t_1, \ldots, t_n)$ which are constructed by repeated application of sums, differences, products, quotients and roots of any index; we will assume in the sequel that $r > n$. More formally, a radical parametrization is a tuple of elements of a radical extension of the field $\mathbb{C}(\overline{t})$ of rational functions in the variables $\overline{t}$. In the following we approach the concept by means of Field Theory.

Definition 2.1. A *radical tower over* $\mathbb{C}(\overline{t})$ is a tower of field extensions $\mathbb{F}_0 = \mathbb{C}(\overline{t}) \subseteq \cdots \subseteq \mathbb{F}_{m-1} \subseteq \mathbb{F}_m$ such that $\mathbb{F}_i = \mathbb{F}_{i-1}(\delta_i) = \mathbb{F}_0(\delta_1, \ldots, \delta_i)$ with $\delta_i^{e_i} = \alpha_i \in \mathbb{F}_{i-1}$, $e_i \in \mathbb{N}$. In particular, $\mathbb{C}(\overline{t})$ is a radical tower over itself.

Definition 2.2. A *radical parametrization* is a tuple $\overline{x}(\overline{t})$ of elements of the last field $\mathbb{F}_m$ of some radical tower over $\mathbb{C}(\overline{t})$, such that their Jacobian has rank n.

Remark 2.3.

(i) A rational parametrization is a radical parametrization.

(ii) The Jacobian is defined by extension of the canonical derivations $\frac{\partial}{\partial t_i}$ from $\mathbb{C}(\overline{t})$ to $\mathbb{F}_m$ (see Lang, 2002, Chapter 8.5, Theorem 5.1, or Zariski and Samuel, 1975, Chapter II.17, Corollary 2 for the formal details). One can calculate the derivatives of the δ's recursively as follows: for each expression $\delta_i^{e_i} = \alpha_i$ in the definition of the tower we write a relation $\Delta_i^{e_i} = \alpha_i(\Delta_1, \ldots, \Delta_{i-1})$ where $\Delta_1, \ldots, \Delta_i$ are new variables dependent on the $\overline{t}$. Then we can differentiate with respect to any t_i to obtain

$$e_i \Delta_i^{e_i - 1} \frac{\partial \Delta_i}{\partial t_j} = \frac{\partial \alpha_i}{\partial t_j},$$

the right hand side involving $\Delta_1, \ldots, \Delta_{i-1}$ and their derivatives. Substituting the δ's into the Δ's we obtain an explicit relation between $\frac{\partial \delta_i}{\partial t_j}$ and the previous partial derivatives.

Let us illustrate the notion of radical parametrization with an example, in order to relate the usual way of writing radical expressions to our Definition 2.2. See for instance Caviness and Fateman (1976) and Davenport et al. (1988, Section 2.6) for more information on the topic of representation and simplification of radical functions.

Example 2.4. One would expect that the expression

$$\left(\frac{1}{\sqrt[6]{t}\sqrt[3]{t} - \sqrt{t}}, t \right) \tag{2.1}$$

is not defined because the denominator is zero. This is due to the default interpretations of the roots as the principal branches (i.e. $\sqrt[n]{1} = +1$). Let us try to be more explicit about those branches by interpreting (2.1) as a parametrization in the sense of Definition 2.2.

For this purpose, we need to introduce a radical tower over $\mathbb{F}_0 = \mathbb{C}(t)$. We can consider the following tower

$$\mathbb{T} := \left[\mathbb{F}_0 \subset \mathbb{F}_0(\delta_1) \subset \mathbb{F}_0(\delta_1, \delta_2) \subset \mathbb{F}_0(\delta_1, \delta_2, \delta_3), \text{ where } \delta_1^2 = t,\ \delta_2^3 = t,\ \delta_3^6 = t \right].$$

Note that there are different choices for the δ_i, but all possible choices of conjugates generate the same tower. Thus, we can write the parametrization as

Please cite this article in press as: Sendra, J.R., et al. Algebraic and algorithmic aspects of radical parametrizations. Comput. Aided Geom. Des. (2017), http://dx.doi.org/10.1016/j.cagd.2017.01.002