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Low rank interpolation of boundary spline curves

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Dedicated to the memory of the late Professor Gerald E. Farin

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ABSTRACT

The coefficients of a tensor-product spline surface in $\mathbb{R}^d$ with $m \times n$ control points form a tensor of order 3 and dimension (m, n, d). Motivated by applications in isogeometric analysis we analyze the *rank* of this tensor. In particular, we propose a new construction for low rank tensor-product spline surfaces from given boundary curves. While the results of this construction are generally not affinely invariant, we propose a simple standardization procedure that guarantees affine invariance for $d = 2$. In addition we provide a detailed comparison with existing constructions of spline surfaces from boundary data.

1. Introduction and overview

The construction of (tensor-product) spline surfaces from four given boundary curves is one of the classical problems in Computer Aided Geometric Design. *Coons interpolation* is a well-established construction of such surfaces (Coons, 1964; Farin, 2001). A linear interpolant is created separately in each of the two parametric directions and a bilinear interpolant of the patch corners is subtracted from their sum. The method is robust and simple.

Farin and Hansford (1999) introduced the discrete version of Coons interpolation as a special instance of using masks. They propose the class of *permanence patches*. These are spline surfaces where the mask generating the interior control points is a linear blend of the Lagrange–Euler mask (corresponding to discrete Coons patch) and the Laplace mask (corresponding to Laplacian smoothing). In the special case of Bézier surfaces, the discrete Coons patch is the same as Coons patch (Farin, 1992; Farin and Hansford, 1999).

In the case of Bézier surfaces, Monterde and Ugail (2004, 2006) choose the inner control points such that the resulting surface satisfies certain fourth-order partial differential equation (see also Jüttler et al., 2006). This also includes Coons interpolation as a special case. Similarly to Coons interpolation, this approach can also be modified to allow for Hermite interpolation (Centella et al., 2009).

Constructions of surfaces from their boundary curves have found new applications in isogeometric analysis, since they are able to generate spline parametrizations for the computational domain of a numerical simulation from a given boundary representation (see Falini et al., 2015, and the references therein). In this context it has been observed that an analysis and subsequent optimization of the *rank* of a parametrization (which will be discussed in more detail in Section 2 of this paper)

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can lead to substantial improvements of the overall efficiency of the numerical simulation (Mantzaflaris et al., 2014). This observation has motivated us to explore interpolation techniques that are able to generate low rank spline surfaces.

In the context of implicit *surfaces*, low rank spline representations were used recently by Pan et al. (2016). Their research was motivated by the need to reduce the memory consumption.

After recalling the concept of tensor rank and adapting to the context of spline surfaces we propose an algorithm for coordinate-wise rank-2 interpolation of boundary curves. We note that the results are not affinely invariant as the algorithm does not commute with affine transformations. For the case of planar data we use a transformation to a reference position in order to restore affine invariance.

We show that the new interpolation method satisfies a permanence principle, which is similar to the case of Coons surfaces (Farin and Hansford, 1999), and that it reproduces bilinear surfaces. Using a series of examples we compare the new method with several existing techniques, which include biharmonic interpolation (Monterde and Ugail, 2004), Laplacian smoothing and Coons interpolation. These examples allow us to conclude that both Coons interpolation and our new method produce low rank parametrizations. It should be noted that the methods discussed in this paper do not provide direct theoretical guarantees for injectivity, since they do not possess properties such as a min–max principle, which is known for harmonic mappings. We refer to Gravesen et al. (2014) for a thorough discussion of the injectivity of planar domain parametrizations.

The remainder of the paper is organized as follows. We first introduce the notation and recall the notion of tensor rank. We then formulate our Algorithm CR2I for coordinate-wise rank-2 interpolation in Section 3, and we discuss the permanence principle. The following section shows how to restore affine invariance in the bivariate case by introducing a standard position. Other approaches to boundary interpolation are summarized in Section 5, and this is followed by an example-based comparison with the new methods. Finally we conclude the paper.

2. Preliminaries

We recall the notion of tensor rank and adapt it to the case of spline surfaces.

2.1. Rank of tensor-product spline surfaces

We consider two univariate B-spline bases

$$\boldsymbol{\beta}(s) = [\beta_i(s)]_{i=1,\dots,m} \quad \text{and} \quad \boldsymbol{\tau}(t) = [\tau_j(t)]_{j=1,\dots,n},$$

which are defined by two open knot vectors with boundary knots 0 and 1. It is not assumed that the degrees of the two spline bases are equal but we assume that $m, n \geq 3$. Recall that their tensor product

$$\boldsymbol{\beta}(s) \otimes \boldsymbol{\tau}(t) = [\beta_i(s)\tau_j(t)]_{i=1,\dots,m;\, j=1,\dots,n}$$

defines the tensor product spline basis. Given a *coefficient tensor*

$$\mathbf{C} = [c_{ijk}]_{i=1,\dots,m;\, j=1,\dots,n; k=1,\dots,d} \in \mathbb{R}^{m\times n\times d}$$

of order 3 and dimension (m, n, d), we define a *tensor product spline surface* in $\mathbb{R}^d$,

$$\mathbf{p}(s,t) = \mathbf{C} : (\boldsymbol{\beta}(s) \otimes \boldsymbol{\tau}(t)) = \left[\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ijk}\beta_i(s)\tau_j(t) \right]_{k=1,\dots,d}, \tag{1}$$

with the parameter domain $[0, 1]^2$. Here we use the symbol **:** to denote the Frobenius product, which compactly expresses the summation with respect to the indices of the tensor-product basis. In the special case $d = 1$ we call $\mathbf{p}$ *scalar-valued spline function* and sometimes write simply p instead of $\mathbf{p}$. If $d = 2$, we call $\mathbf{p}$ a *planar parametrization*, because if $\mathbf{p}$ is bijective, it parametrizes the domain enclosed by the planar boundary curves.

Any tensor of order 3 admits a representation as a finite sum of tensor-products of vector triplets,

$$\mathbf{C} = \sum_{r=1}^{R} \mathbf{u}^r \otimes \mathbf{v}^r \otimes \mathbf{w}^r, \quad \text{or, equivalently,} \quad c_{ijk} = \sum_{r=1}^{R} u_i^r v_j^r w_k^r \tag{2}$$

for some vectors

$$\mathbf{u}^r = [u_i^r]_{i=1,\dots,m} \in \mathbb{R}^m, \quad \mathbf{v}^r = [v_j^r]_{j=1,\dots,n} \in \mathbb{R}^n \quad \text{and} \quad \mathbf{w}^r = [w_k^r]_{k=1,\dots,d} \in \mathbb{R}^d.$$

Clearly, each tensor possesses infinitely many representations of this form. The minimum number R for all possible representations is called the *tensor rank* of $\mathbf{C}$. In particular, the null tensor has rank 0.

This representation was introduced in order to address the "curse of dimension" concerning the memory requirements, especially for higher order tensors. Storing the representation (2) requires $O(R(m+n+d))$ memory, while the full coefficient tensor has mnd elements. Using the representation (2) is advantageous if the tensor rank satisfies $R \ll \min\{m, n\}$.

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