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Greville abscissae of totally positive bases [☆]

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ABSTRACT

For a given totally positive space of continuous functions, we analyze the construction of totally positive bases of the space of antiderivatives. If the functions of the totally positive space have continuous derivatives, normalization properties can be used to describe totally positive bases of the space of derivatives and relate them with properties of the Greville abscissae.

1. Introduction

Integral recurrence formulae for B-splines have been often used in the past. The definition of B-spline as a divided difference of a truncated power function and the Hermite–Gennochi formula lead to integral recursions. One of the first papers where it is observed that the sequence of B-spline bases can be obtained by successive integration in the general context of Chebyshevian splines is Bister and Prautzsch (1997). Bernstein polynomials as well as many other examples of totally positive bases in extended Chebyshev spaces are included in this setting.

Totally positive bases (TP) are bases whose collocation matrices have nonnegative minors. This kind of bases are commonly used in computer-aided design due to their shape preserving properties (see Goodman, 1996). Among all normalized TP bases of a space, we can find normalized B-bases, which are the optimal shape preserving bases (cf. Carnicer and Peña, 1994).

Spaces containing algebraic polynomials and trigonometric or hyperbolic functions have attracted much interest in the field of computer-aided geometric design (Zhang, 1996; Mainar et al., 2001). In Chen and Wang (2003), integral constructions of Bernstein-like basis for cycloidal spaces

$$C_n = \langle \cos t, \sin t, 1, t, \dots, t^{n-2} \rangle$$

have been provided. In Costantini et al. (2005), such constructions are discussed in a more general setting, showing that the integral constructions provide TP bases. In particular, the normalized B-basis is expressed using integrals of a B-basis of the space of derivatives.

Greville abscissae are the coefficients of the function t with respect to a given basis and play a fundamental role in the definition of Bernstein-like operators in spaces of exponential polynomials (cf. Aldaz et al., 2009).

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In this paper, the construction of TP bases for the space of antiderivatives of a given space of continuous functions with a TP basis is analyzed. We also describe TP bases of the space of derivatives, whose normalization is related with properties of the Greville abscissae.

In contrast to other approaches, we do not require that the spaces with TP bases are extended Chebyshev or piecewise extendend Chebyshev spaces. In Mazure (2009), similar problems on integral constructions, derivative spaces and Greville abscissae are analyzed with powerful techniques, under the hypothesis that the space of derivatives is extended Chebyshev. In Section 7 of Mazure (2009), the question of extending the results to a more general context is explored. In Mazure (2011), piecewise Chebyshev spaces are analyzed using knot insertion techniques. In our approach, we deal not only with the normalized B-basis and we show that the integral or derivative constructions can be applied to the more general class of TP bases.

In Section 2, we describe integral constructions of normalized TP bases and normalized B-bases. The construction of TP bases and B-bases of the space of derivatives is presented in Section 3. In Section 4, we consider shape preserving representations of curves and we obtain a derivative formula of the curve involving the Greville abscissae. This formula relates the normalized B-basis of a given space with that of the space of derivatives. We also include examples illustrating that the integral constructions are valid even when the starting space is not extended Chebyshev. In Section 5, we show the equivalence of the existence of a normalized TP basis in the space of derivatives with the fact that Greville abscissae of shape preserving representations with the endpoint interpolation property are increasing. In Section 6, we present some applications, including sufficient conditions for Bernstein-like operators to be convexity preserving. Finally, we obtain a generalization of Theorem 25 of Aldaz et al. (2009) for general cycloidal spaces, deriving conditions on the length of the interval domain to ensure that the Greville abscissae of the normalized B-basis (corresponding to the nodes of the associated Bernstein operator) are strictly increasing.

2. Integral constructions with totally positive bases

Let us denote by $D : f \in C^1[a,b] \mapsto f' \in C[a,b]$ the derivative operator. For a given space of functions $U \subset C[a,b]$, we introduce the space

$$D^{-1}U := \{v \in C^1[a,b] \mid v' \in U\}.$$

Observe that $\ker D$ is the one dimensional space of constant functions. Hence, if $\dim U = n$, then $D^{-1}U$ contains the constant functions and $\dim D^{-1}U = n+1$.

Definition 1. A matrix is *totally positive* (TP) if all its minors are nonnegative. A system of functions $(u_0,\dots,u_n)$ defined on the subset $I \subseteq \mathbf{R}$ is *totally positive* (TP) if all its collocation matrices

$$M\begin{pmatrix} u_0,\dots,u_n \\ t_0,\dots,t_n \end{pmatrix} := (u_j(t_i))_{i,j=0,\dots,n}, \quad t_0 < \cdots < t_n \text{ in } I$$

are TP. A TP system of functions on I is *normalized* (NTP) if $\sum_{i=0}^n u_i(t) = 1$, for all $t \in I$.

In the following result we show how to construct a TP system of functions in $D^{-1}U$, starting from a TP system of functions in U.

Proposition 2. *Let U be an n-dimensional subspace of $C[a,b]$. If $(u_0,\dots,u_{n-1})$ is a TP system of functions in U, then the system $(f_0,\dots,f_n)$ defined by*

$$f_0(t) := 1, \quad f_i(t) := \int_a^t u_{i-1}(x)dx, \quad i = 1,\dots,n, \quad t \in [a,b], \tag{1}$$

is TP. Moreover, if $(u_0,\dots,u_{n-1})$ is a TP basis of U, then $(f_0,\dots,f_n)$ is a TP basis of $D^{-1}U$.

Proof. In order to prove the total positivity of $(f_0,\dots,f_n)$, it is sufficient to show that

$$d := \det M\begin{pmatrix} f_{i_0},\dots,f_{i_k} \\ t_0,\dots,t_k \end{pmatrix} \ge 0,$$

for every $i_0 < \cdots < i_k$ in $\{0,\dots,n\}$ and all $t_0 < \cdots < t_k$ in I. First let us analyze the case $i_0 \ne 0$. Subtracting to each row of the matrix $M\begin{pmatrix} f_{i_0},\dots,f_{i_k} \\ t_0,\dots,t_k \end{pmatrix}$ the previous one, taking into account the multilinearity of the determinant and the total positivity of $(u_0,\dots,u_{n-1})$, we deduce that

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