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Complexity of hierarchical refinement for a class of admissible mesh configurations

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ABSTRACT

An adaptive isogeometric method based on d-variate hierarchical spline constructions can be derived by considering a refine module that preserves a certain class of admissibility between two consecutive steps of the adaptive loop (Buffa and Giannelli, 2016). In this paper we provide a complexity estimate, i.e., an estimate on how the number of mesh elements grows with respect to the number of elements that are marked for refinement by the adaptive strategy. Our estimate is in the line of the similar ones proved in the context of adaptive finite element methods.

1. Introduction

Throughout the last years, isogeometric methods have gained widespread interest and are a very active field of research (Cottrell et al., 2009; Beirão da Veiga et al., 2014) investigating a wide range of applications and theoretical questions. Due to the tensor-product structure of splines, there exist very stable procedures to perform mesh refinement and degree raising which are known in the literature as h-refinement, p-refinement, k-refinement (Cottrell et al., 2009). While these algorithms are very efficient, the preservation of the tensor-product structure at least locally on each patch, produces a dramatic increase of degrees of freedom together with elongated elements. Mainly for this reason, several approaches have been proposed to alleviate these constraints and they all need the definition of B-splines over non-tensor-product meshes. Indeed, there are several strategies and we mention here T-splines (Bazilevs et al., 2010), hierarchical B-splines (Forsey and Bartels, 1988; Kraft, 1997; Kuru et al., 2014) and THB-splines (Giannelli et al., 2012), but also LR splines (Dokken et al., 2013; Bressan, 2013), hierarchical T-splines (Evans et al., 2015), modified T-splines (Kang et al., 2013), PHT-splines (Deng et al., 2008; Wang et al., 2011) amongst others.

Clearly, the development of adaptive strategies exploiting the potential of non-tensor-product splines is an interesting and important step which has been approached in a number of papers, at least from the practical point of view. In fact, despite their performance in experiments (Bazilevs et al., 2010; Dörfel et al., 2010; Beirão da Veiga et al., 2014; Kuru et al., 2014; Evans et al., 2015), the advantages of mesh-adaptive isogeometric methods have not been assessed in theory until today. Partial results on approximation, efficient and reliable error estimates and convergence of the adaptive procedure, have been proven in preliminary work (Buffa and Giannelli, 2016) in the context of (truncated) hierarchical splines. We aim to

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continue this study and to provide further ingredients that are needed for a proof of optimal convergence of the proposed adaptive approach in the spirit of adaptive finite element methods (Binev et al., 2004; Stevenson, 2007; Cascón et al., 2008; Carstensen et al., 2014).

In particular, in this paper we address the complexity of the mesh refinement procedure proposed in (Buffa and Giannelli, 2016). The relation between the set of marked elements and the overall number of refined elements introduced by the refine module is not straightforward: additional elements may be refined to create only (strictly) *admissible* meshes. Admissibility is a restriction to suitably graded meshes that allows for the adaptivity analysis of hierarchical isogeometric methods. Consequently, in order to define an automatic strategy to steer the adaptive method, the refinement is recursively propagated over a *suitable neighborhood* of any marked element. By starting from an initial mesh configuration $\mathcal{Q}_0$, let $\{\mathcal{Q}_j, \mathcal{M}_j\}_{j\geq 0}$ be the sequence of meshes $\mathcal{Q}_j$ and marked elements $\mathcal{M}_j$ computed by the adaptive scheme. At step j of the refinement loop, the adaptive algorithm refines the marked subset of elements $\mathcal{M}_{j-1} \subseteq \mathcal{Q}_{j-1}$, together with some additional ones, to obtain the refined mesh $\mathcal{Q}_j$ with the same properties as $\mathcal{Q}_{j-1}$. A complexity estimate of the form

$$\#\mathcal{Q}_J - \#\mathcal{Q}_0 \leq \Lambda \sum_{j=0}^{J-1} \#\mathcal{M}_j \,, \tag{1}$$

with some positive constant Λ, provides a bound for the ratio of the newly inserted elements $\#\mathcal{Q}_J - \#\mathcal{Q}_0$ introduced up to step J and the cumulative number $\sum_{j=0}^{J-1} \#\mathcal{M}_j$ of elements marked for refinement at each intermediate step in the subdivision process that leads from the initial to the final mesh. This allows to control the propagation of the refinement beyond the set of elements initially selected by the marking strategy. Our main result provides a complexity estimate (1) for an adaptive isogeometric method based on d-variate hierarchical spline constructions of any degree. An analogous complexity analysis is currently available for bivariate and trivariate T-splines (Morgenstern and Peterseim, 2015; Morgenstern, submitted for publication).

This paper is organized as follows. In Section 2, we recall notation and basic results from (Buffa and Giannelli, 2016). Section 3 is devoted to the announced complexity estimate. Conclusions and an outlook to future work are given in Section 4.

2. Hierarchical refinement

In this section, we recall some notation and basic results from (Buffa and Giannelli, 2016). Since the complexity analysis of the REFINE module can be performed directly in the parametric setting, we avoid to introduce the two different notations for parametric/physical domains.

2.1. The truncated hierarchical basis

Let $V^0 \subset V^1 \subset \cdots \subset V^{N-1}$ be a nested sequence of tensor-product d-variate spline spaces of fixed degree $\mathbf{p} = (p_1, \ldots, p_d)$ defined on a closed hypercube D in $\mathbb{R}^d$. For each level ℓ, with $\ell = 0, 1, \ldots, N-1$, we denote by $\mathcal{B}^\ell$ the normalized tensor-product B-spline basis of the spline space V^ℓ defined on d knot sequences $T_1^\ell, \ldots, T_d^\ell$, for $\ell = 0, \ldots, N-1$, containing the different knot values in any coordinate direction. Let $\mu(T_i, t)$ be the multiplicity of t in T_i, where $0 \leq \mu(T_i, t) \leq p_i + 1$ and $\mu(T_i, t) = 0$ if t is not a knot in T_i. In order to define nested spaces, the knot sequences are also assumed to be nested, namely $\mu(T_i^{\ell+1}, t) \geq \mu(T_i^\ell, t)$.

Each space V^ℓ has an associated grid G^ℓ consisting of axis-aligned boxes such that the restriction of a function that belongs to V^ℓ to any of these cells is a tensor-product polynomial of degree $\mathbf{p}$, and G^ℓ is the coarsest grid with that property. We assume that G^0 consists of open hypercubes with side length 1. The Cartesian product of d open intervals between adjacent (and non-coincident) grid values defines a quadrilateral element Q of G^ℓ. For all $Q \in G^k$ we denote by $h_Q := 2^{-k}$ the length of its side, and by $\ell(Q)$ its level, i.e., $\ell(Q) = k$.

Remark 1. The analysis could be generalized to the more general case of a non-uniform initial knot configuration (by suitably taking into account the corresponding maximum local mesh size).

In order to define hierarchical spline spaces, we consider a nested sequence of closed subdomains $\Omega^0 \supseteq \Omega^1 \supseteq \ldots \supseteq \Omega^{N-1}$ of D. Any Ω^ℓ is the union of the closure of elements that belong to the tensor-product grid of the previous level. The *hierarchical mesh* $\mathcal{Q}$ is defined as

$$\mathcal{Q} := \left\{ Q \in \mathcal{G}^\ell, \, \ell = 0, \ldots, N-1 \right\}, \tag{2}$$

where

$$\mathcal{G}^\ell := \left\{ Q \in G^\ell : Q \subset \Omega^\ell \wedge Q \not\subset \Omega^{\ell+1} \right\} \tag{3}$$

is the set of active elements of level ℓ. Fig. 1 shows two hierarchical meshes related to the case $d = 1$ and $d = 2$, respectively.

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