ARTICLE IN PRESS
JID: CAG [m5G;June 20, 2017;21:20]

Computers & Graphics 000 (2017) 1–11

Contents lists available at ScienceDirect

Computers & Graphics

journal homepage: www.elsevier.com/locate/cag

Special Issue on SMI 2017

Hierarchical Forman Triangulation: A multiscale model for scalar field analysis

Federico Iuricich*, Leila De Floriani

Center for Geospatial Information Science, University of Maryland, College Park, MD, USA

ARTICLE INFO

Article history:
Received 4 April 2017
Revised 17 May 2017
Accepted 25 May 2017
Available online xxx

Keywords:
Topological data analysis
Simplification
Multi-scale models
Discrete Morse theory

ABSTRACT

We consider the problem of analyzing the topology of scalar fields defined on a triangulated shape by using a multi-scale approach, which allows reducing storage costs and computation times, and supports interactive inspection and classification of topological features. We define and implement a multi-scale model that we call a *Hierarchical Forman Triangulation (HFT)*, where a 3D shape or a terrain is discretized as a triangle mesh, and its topology is described by defining a discrete Morse gradient field based on function values given at the vertices of the mesh. We introduce a new edge contraction operator, which does not change the behavior of the gradient flow and does not create new critical points, and we apply it in combination with a topological simplification operator which eliminates critical elements in pair. By combining the two operators in a sequence, we generate the *HFT*. We discuss and implement a compact encoding for the *HFT* that has a lower storage cost with respect to the triangle mesh at full resolution. We show the effectiveness of this new hierarchical model by extracting representations of terrains and shapes endowed with a scalar field at different, uniform and variable, scales and by efficiently computing topological features and segmentations.

1. Introduction

Computational topology is a rapidly developing field in data and shape analysis. It is used to support classification and understanding combined with machine learning techniques [1] and as the basis for interactive analysis and inspection through visualization [2]. Topological tools are rooted in Morse theory and persistent homology. This latter has produced shape signatures, like the barcode or the persistent diagram, while the former has been the basis for extracting topological features, like the Reeb graph, which describes the evolution of the level sets of a scalar function defined on a manifold shape, or the Morse and Morse–Smale complexes, which provide a segmentation of a shape induced by the regions of influence of the critical points of a scalar function defined on it [3]. These latter have been extensively used for terrain analysis [4], shape analysis [5,6] or remeshing [7].

The purpose of our work is extracting Morse features, like the ascending and descending manifolds forming the Morse complexes, efficiently and effectively. We consider a terrain or a 3D shape discretized through a triangle mesh, with a scalar value associated with its vertices. Because of the large size of the meshes, most of the recent approaches to extract topological features from data are based on a discrete version of Morse theory for cell and simplicial complexes [8], which allows an entirely combinatorial and derivative-free approach to Morse feature computation.

Since data are affected by noise, many spurious critical points and cells can be generated. Simplification approaches have been defined for dealing with both noise and data redundancy [9,10]. Simplifying a scalar field using topological simplifications means canceling critical points in pairs, thus reducing the number of cells in the Morse complexes. Simplification approaches have been recognized as effective but not efficient, especially to support data inspection and understanding through visualization. Multi-resolution approaches have been introduced for providing faster interactions and more degrees of freedom on the extracted representations [11–13]. Most of these models interact with the morphology but leave the underlying mesh untouched. This is a serious issue when working with big datasets since the complexity of extracting, representing and visualizing Morse features is mainly affected by the resolution of the mesh and not by the size of the Morse complexes.

Our work is inspired by Iuricich and De Floriani [14] where the authors introduce the first edge contraction operator for a triangle mesh endowed with a scalar field which maintains the Forman gradient. The operator is used to create a progressive model, consisting of a sequence of simplifications of the mesh and of the Forman gradient. On the other hand, we define a multi-scale model for a triangulated shape endowed with a scalar field, that we call a *Hierarchical Forman Triangulation (HFT)*. The HFT allows

* Corresponding author.
E-mail address: iurif@umd.edu (F. Iuricich).

http://dx.doi.org/10.1016/j.cag.2017.05.015
0097-8493/

extracting both a mesh and a topological representation at different levels of topological and geometric resolutions. The model is generated based on two simplification operators: the *edge contraction* operator, which simplifies the mesh, and the *cancellation* operator, which simplifies the morphology. The edge contraction operator that we will introduce in Section 5 is an improvement over the one defined in [14] since it imposes conditions on the Forman gradient V only. Given a triangle mesh Σ endowed with a scalar function given at its vertices, we build a discrete Morse gradient V on Σ compatible with the scalar field; then, through *edge contractions* and *cancellations*, we simplify both Σ and V. The *HFT* is built from the resulting sequence of simplifications. The inverse of the atomic simplifications used to generate the *HFT* together with a partial order dependency relation between pairs of simplifications form the structure of the *HFT*, from which representations at different scales, also variable across the mesh, can be efficiently extracted. The major contributions of this work are the definition and implementation of:

- a simplification operator for triangle meshes endowed with a Forman gradient, which does not eliminate or generate critical elements, and generalizes the operator presented in [14];
- a new refinement operator, inverse of the latter, that operates on the mesh and on the Forman gradient without creating or eliminating critical elements;
- a new multi-scale model, the Hierarchical Forman Triangulation (*HFT*), which combines mesh and topological updates, based on discrete Morse theory;
- a dependency relation between topological updates, which is minimal in the number of dependencies required, thus greatly enhancing the expressive power of the multi-scale model.

The *HFT* has a low storage cost, lower than that of the mesh at full resolution, and provides a high flexibility in adjusting the resolution of the mesh to comply with the scale of the topological representation. This allows extracting variable-scale representations of the mesh endowed with the gradient as well as multi-scale Morse features efficiently.

2. Background notions

Morse theory [15,16] is a mathematical tool studying the relationships between the topology of a manifold shape M and the critical points of a smooth scalar function f defined over M. Piecewise-linear Morse theory transposes some results from Morse theory to piece-wise linear functions [17]. Here, we focus on a combinatorial counterpart of Morse theory for cell complexes due to Forman and called *Discrete Morse Theory (DMT)* [8]. Since simplicial complexes are common discretization structures [18] for shapes in low and high dimensions, we review *DMT* focusing only on these latter. Recall that a *k-dimensional simplex*, or simply a *k-simplex*, σ is the convex hull of $k+1$ geometrically independent points in $\mathbb{R}^n$. 0-, 1- and 2-simplices are also called *vertices*, *edges*, and *triangles*, respectively. A *triangle mesh* is a special case of a simplicial complex: it is formed by vertices, edges and triangles and has a manifold domain.

We consider a pair (Σ, F), where Σ is a triangle mesh and $F : \Sigma \to \mathbb{R}$ is a scalar function defined on all the simplices of Σ. Function F is a *discrete Morse function* (also called a *Forman function*) if and only if, for every k-simplex $\sigma \in \Sigma$, all the $(k-1)$-simplices on the boundary of σ have a lower function value than σ, and all the $(k+1)$-simplices bounded by σ have a higher function value than σ, with at most one exception. If there is such an exception, it defines a pairing of cells, called a *gradient pair*. A gradient pair can be viewed as an *arrow* in which the head is a k-simplex and the tail a $(k-1)$-simplex. A simplex that is not a head or a tail of any arrow is a *critical simplex*. A *V-path* is a sequence of simplices $[\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \sigma_i, \tau_i, \ldots, \sigma_q, \tau_q]$ such that σ_i and σ_{i+1} are on the boundary of τ_i and (σ_i, τ_i) are paired simplices, where $i = 0, \ldots, q$.

The collection of all paired and critical simplices of Σ forms a *discrete Morse gradient* (also called a *Forman gradient*) if there are no closed V-paths, i.e., if all V-paths are acyclic. In Fig. 1(a)) a Forman gradient is shown: it has two critical triangles (t and t_1), one critical edge e, and one critical vertex v. Given a triangle mesh Σ endowed with a scalar function f defined on its vertices, we can always compute a Forman gradient V without computing the Forman function F explicitly. In our work we compute the Forman gradient through the algorithm in [19]. In the following, we denote a triangle mesh Σ endowed with a Forman gradient V as a pair (Σ, V). We call a *separatrix V_j-path* any V-path of the following form: $[\tau, \sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \sigma_i, \tau_i, \ldots, \sigma_q, \tau_q, \sigma]$, where τ and σ are two critical simplices of dimension $j+1$ and j, respectively. Thus, in a triangle mesh Σ we will have separatrix V_0-paths connecting a critical edge to a critical vertex and separatrix V_1-paths connecting a critical triangle to a critical edge (see Fig. 1).

Topological features are defined in the discrete case in terms of the Forman gradient and its paths. The *critical net* consists of the critical vertices, edges, and triangles plus the separatrix V_0- and V_1-paths connecting them. Any descending k-manifold (which is a k-cell of the descending Morse complex), is the collection of the k-simplices of Σ reached by the gradient paths starting from critical k-simplex. Dually, an ascending k-manifold (a k-cell of the ascending Morse complex) is the collection of the $(2-k)$-simplices reached by the gradient paths (visited backward) starting from a critical $(2-k)$-simplex. Fig. 1(e) illustrates the descending 2-manifold associated with triangle t. A description of the algorithms used for extracting the ascending and descending manifolds from a Forman gradient can be found in Section 8.

3. Related work

Morse complexes can be simplified by applying an operator defined in smooth Morse theory, called *cancellation* [16]. A cancellation removes two critical points of consecutive index which are connected by a separatrix line. This operator has been investigated in 2D [11,20–22] and 3D [10,23], by considering piecewise-linear shape approximations.

Given a sequence of cancellations, a hierarchical model is built by organizing into a hierarchy the refinements which are inverse of such cancellations, each refinement (also called anti-cancellation) performing an undo of the corresponding cancellation. Several hierarchical models exist for representing the morphology of a triangle mesh endowed with a scalar field [3]. They can be classified as: *progressive models*, that just represent the sequence of refinements reversing the cancellation sequence, and *multi-resolution models*, that organize the refinements according to a partial order relation of mutual dependencies among the refinements. These latter have a much higher expressive power, since they support the extraction of a large number of representations not encountered during the cancellation process. The first progressive morphological model has been developed in the context of image analysis. The hierarchical approach described in [24] defines a containment hierarchy for the regions of the watershed segmentation computed on the image. In [20] a progressive model is defined for the morphology of a terrain. The hierarchy is created by applying cancellations on the critical net. The hierarchy is encoded as the critical net at the coarsest resolution plus the sequence of *anti-cancellations*, inverse to the cancellations used in the construction phase.

In [4,11] a *dependency relation* among anti-cancellations is introduced for building a multi-resolution model. In [11], the dependency relation between two refinements is defined in terms of a *diamond*. The diamond associated with an anti-cancellation(q, p)