Int. J. Electron. Commun. (AEÜ) 78 (2017) 220–227

Contents lists available at ScienceDirect

International Journal of Electronics and Communications (AEÜ)

journal homepage: www.elsevier.com/locate/aeue

Regular paper

A simple three-dimensional fractional-order chaotic system without equilibrium: Dynamics, circuitry implementation, chaos control and synchronization

Viet-Thanh Pham [a,e,*], Sifeu Takougang Kingni [b], Christos Volos [c], Sajad Jafari [d], Tomasz Kapitaniak [e]

[a] School of Electronics and Telecommunications, Hanoi University of Science and Technology, 01 Dai Co Viet, Hanoi, Viet Nam
[b] Department of Mechanical and Electrical Engineering, Institute of Mines and Petroleum Industries, University of Maroua, P.O. Box 46, Maroua, Cameroon
[c] Department of Physics, Aristotle University of Thessaloniki, Thessaloniki GR-54124, Greece
[d] Biomedical Engineering Department, Amirkabir University of Technology, Tehran 15875-4413, Iran
[e] Division of Dynamics, Lodz University of Technology, Stefanowskiego 1/15, 90-924 Lodz, Poland

ARTICLE INFO

Article history:
Received 30 January 2017
Accepted 14 April 2017

Keywords:
Chaos
Equilibrium
Fractional order
Circuit
Control
Synchronization

ABSTRACT

Chaotic systems without equilibrium points have been investigated and received significant attention recently. In this work, we propose a new three-dimensional fractional-order chaotic system without equilibrium. Dynamics of the fractional-order system are investigated and a circuit implementation of the system by using electronic components is presented. Interestingly, bistable chaotic attractors of such fractional system are discovered. In addition, we are able to control and synchronize the system by using active control and unidirectional coupling.

1. Introduction

Fractional calculus has been studied since the 17th century and its applications have been reported in various areas ranging from physics, electrical circuit, chemical engineering, electromagnetic theory to control [1–13]. It is interesting that chaotic behavior has been observed in fractional-order systems [14–17]. Due to the complexity and the advantages of fractional derivatives, engineering applications using fractional-order chaotic systems have been also developed [18–21]. Different chaotic fractional-order systems have been presented in the literature such as fractional Chua system [14–16], fractional Lorenz system [17], fractional Rössler system [22], fractional Chen system [23–25], fractional Lü system [26], fractional Duffing system [27], fractional incommensurate order financial system [28], fractional order switching system [29] and so on. It is noted that there are countable numbers of equilibrium points in these chaotic fractional-order systems.

Recently, researchers have shown an increased interest in fractional systems without equilibrium exhibiting chaotic behavior [30]. Despite of the absence of equilibrium points, such systems display complex behavior. Moreover, they are different from other previous ones especially from the view point of computation [31–33]. Their attractors, called "hidden attractors", cannot be localized straight forwardly by applying a standard computational procedure [34–36]. No-equilibrium systems with hidden attractors have attracted interest in the last few years [37–41].

Reported fractional-order chaotic systems without equilibrium and their noticeable features are summarized in Table 1. The first non-equilibrium fractional-order chaotic system was introduced by Li et al. [30]. Li system is a four dimensional system which can generate chaos for the order as low as 3.28 [30]. Zhou and Huang constructed another 4-D fractional-order system without equilibrium point [42]. Zhou-Huang system is non chaotic for its integer-order but it can exhibit chaotic attractor for its fractional-order as low as 3.2. In addition, based on a integer-order hyperchaotic system, authors proposed a hyperchaotic fractional-order system without equilibria [43]. Its hyperchaos exists for the fractional-order as low as 3.84. Rajagopal et al. implemented 4-D fractional-order no equilibrium cubic nonlinear resistor system in FPGA [44]. A natural question is posed "Is there a 3-D fractional-order chaotic system without equilibrium?" Until now a few answers to this question are reported. Only a 3-D fractional

* Corresponding author at: School of Electronics and Telecommunications, Hanoi University of Science and Technology, 01 Dai Co Viet, Hanoi, Viet Nam.
E-mail addresses: pvt3010@gmail.com, viet-thanh.pham@p.lodz.pl (V.-T. Pham).

http://dx.doi.org/10.1016/j.aeue.2017.04.012
1434-8411/

Table 1
Reported fractional-order chaotic systems without equilibrium.

Systems	Dimension	Fractional order	Circuit implementation
[44]	4	3.984	Yes
[43]	4	3.84	No
[30]	4	3.28	Yes
[42]	4	3.2	Yes
[45,46]	3	2.94	No
This work	3	2.7	Yes

system without equilibrium points was investigated by Cafagna and Grassi [45,46]. The presence of chaos in Cafagna–Grassi fractional system was obtained when its fractional-order is as low as 2.94. However it is trivial to verify that this system's fractional-order is close to the integer-order 3. Moreover, the feasibility of such 3-D Cafagna–Grassi fractional system has not been considered. Further studies need to be carried out in order to provide insights for 3-D fractional-order systems without equilibrium.

This study makes a contribution to researches on fractional chaotic systems by discovering a novel fractional-order chaotic system without equilibrium. The description of the no-equilibrium system is presented in the next section while its dynamics are reported in Section 3. Circuit implementation of the theoretical fractional-order chaotic system is studied in Section 4. The abilities of control and synchronization of the fractional system are investigated through active control and unidirectional coupling in Section 5. Finally, we draw conclusion remarks in the last section.

2. Model of the fractional-order chaotic system without equilibrium

Previous studies have presented different definitions of fractional-order derivative, however Grunwald–Letnikov, Riemann–Liouville and Caputo definitions are commonly used [1,47–49]. In this work, we utilized the Caputo definition, which is defined by

$$ {}_0D_t^q f(t) = \begin{cases} \frac{1}{\Gamma(m-q)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{q+1-m}} d\tau, & m-1 < q < m \\ \frac{d^m}{dt^m} f(t), & q = m \end{cases} \tag{1} $$

In Caputo definition (1), m is the first integer which is not less than q ($m = \lceil q \rceil$) while Γ is the Gamma function:

$$ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{2} $$

In this work, we consider a three dimensional fractional-order system given by

$$ \begin{cases} D^q x = y \\ D^q y = -x - yz \\ D^q z = xy + ax^2 - b \end{cases} \tag{3} $$

in which three state variables are x, y, z while two positive parameters are a and b ($a, b > 0$). D^q denotes the Caputo fractional derivative (1) with initial time $t_0 = 0$ and $q \in (0, 1)$. In system (3), the commensurate fractional order is denoted as q. Because system (3) is a three dimensional system, its fractional order is therefore $3q$. It is easy to verify that the fractional-order system (3) has no any equilibrium. It is noted that we have constructed system (3) based on the system NE_8 listed in [39]. Fractional-order system (3) is different from the reported fractional systems [17,22], which have a countable number of equilibrium points. Interestingly, system (3) belongs to a new class of systems without equilibrium [45,46].

The fractional-order system (3) is invariant under the coordinate transformation $(x, y, z) \to (-x, -y, z)$. In other words, there is a rotational symmetry with respect to the z-axis in fractional-order system (3).

It is interesting that the fractional-order system can generate chaotic behavior although there is the absence of equilibrium. For example, chaotic phase portraits of fractional-order system (3) are presented in Fig. 1, for $q = 0.9,\ a = 1.5,\ b = 1.3$, and initial conditions $(x(0), y(0), z(0)) = (0, 0.1, 0)$. In order to calculate the largest Lyapunov exponent of fractional-order system (3) we have applied the practical method reported in [50]. In this case, the largest Lyapunov exponent of the fractional system is 0.1374.

3. Dynamics of the fractional-order chaotic system without equilibrium

We investigate the dynamics of fractional-order system (3) by varying the bifurcation parameter b for $a = 1.5$ and the commensurate fractional order $q = 0.9$. The bifurcation diagram of fractional-order system (3) is shown in Fig. 2. In this work, we have used the Adams-Bashforth-Moulton method to solve fractional differential equations [51]. When decreasing the value of the parameter b from 1.4 to 1.2, there is the presence of a route from non-chaotic behavior to chaos. System generates non-chaotic behavior, as illustrated in Fig. 3. It is noted that obtained results agree with the reported studies about periodic solutions in fractional order systems [46,52–55]. Dynamics of fractional-order system (3) when considering the fractional order q as bifurcation parameter has been presented in Fig. 4. As shown in Fig. 4, the system exhibits non-chaotic and chaotic behavior.

As it has been mentioned in Section 2, there is a rotational symmetry with respect to the z-axis in fractional-order system (3). Thus, we may observe bistable attractors in fractional system (3). Remarkably, we have found the bistable chaotic attractors of the fractional-order system (3) for $q = 0.9,\ a = 1.5,\ b = 1.32$ as illustrated in Fig. 5. To the best of our knowledge, there is no similar result in reported fractional-order systems without equilibrium.

4. Circuit implementation

It is now well established from a variety of studies that circuit implementations of theoretical chaotic models play vital roles in engineering applications [56–59]. Moreover, the feasibilities of theoretical models are verified via their circuit implementations [60]. The circuit implementation of the introduced fractional system without equilibrium is, therefore, discussed in this section.

Previous researches have established that integer chaotic systems have been implemented by analog approach or digital approach [61–63]. Analog realizations of integer chaotic systems consist of discrete active devices or mixed-mode active devices [64–67]. The designers do not worry about the amplitudes of signals because integrated circuit implementations are possible when scaling or normalizing the amptitudes of the magnitudes of the state variables [68,69]. Digital realizations of integer chaotic systems are based on microcontroller or field-programmable gate arrays (FPGA) [70–72]. Recently FPGAs are quite useful for chaotic systems implementation [73]. Similarly, such approaches have been performed for fractional chaotic systems [30,44,74,75].

As can be seen in Fig. 1, the amplitudes of the state variables are smaller that 10. The amplitudes of variables in system fractional-order system (3) are in the range of the output of the operational amplifier. Therefore, we do not have to scale or normalize the amplitudes of the magnitudes of the state variables. It is convenient to implement the proposed fractional-order system (3) with the operational amplifier approach [60,76]. The schematic of