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Linear and sigmoidal fuzzy cognitive maps: An analysis of fixed points

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ABSTRACT

Fuzzy cognitive mapping is commonly used as a participatory modelling technique whereby stakeholders create a semi-quantitative model of a system of interest. This model is often turned into an iterative map, which should (ideally) have a unique stable fixed point. Several methods of doing this have been used in the literature but little attention has been paid to differences in output such different approaches produce, or whether there is indeed a unique stable fixed point. In this paper, we seek to highlight and address some of these issues. In particular we state conditions under which the ordering of the variables at stable fixed points of the linear fuzzy cognitive map (iterated to) is unique. Also, we state a condition (and an explicit bound on a parameter) under which a sigmoidal fuzzy cognitive map is guaranteed to have a unique fixed point, which is stable. These generic results suggest ways to refine the methodology of fuzzy cognitive mapping. We highlight how they were used in an ongoing case study of the shift towards a bio-based economy in the Humber region of the UK.

1. Introduction

Fuzzy cognitive mapping is a method for representing a system based on expert knowledge as a directed weighted graph, with weights chosen from a set, resulting in a semi-quantitative model of the system. This method was first developed by Kosko [15] to better understand systems with numerous interconnections between important components but relatively scarce quantifiable information about the form of these interconnections.

Fuzzy cognitive maps (FCMs) have since been used to model a huge variety of systems, from a heat exchanger [26] to predicting prostate cancer [7]. However the way they are used has developed differently amongst different communities of researchers. These separate development paths can loosely be associated with the community of engineers and biologists, and the community of social scientists. In the former it is possible to know what the system should look like on some level, and thus to provide some quantitative data to fit the model to. This allows learning algorithms to be used in order to improve the model [7,16,12,21,22]. It may also be possible to provide direct feedback from the model into the system it represents to allow the model to be used to directly control the system [16,26]. This has been studied more for general fuzzy models than for FCMs. For instance with appropriate learning algorithms fuzzy models can be used in stabilising an inverted pendulum [10] or in controlling the effect of earthquakes on structures [4,30]. In the social sciences however such direct relationships between the model and the system it represents are not possible, meaning that learning algorithms are rarely applied. In fact even representing the system has its complexities as any representation will be a subjective interpretation of the system. In these disciplines FCMs have developed as an explanatory tool, as a relatively simple way of representing a complex system. From such a model a narrative based analysis can be used to explain how the system might react to certain changes [11,14,18,23,28]. Another way in which the use of FCMs has developed in such subjective fields has been to incorporate the knowledge and insight of those who are enmeshed in the system. That is, the FCM has developed into a participatory modelling methodology where the intersubjective knowledge of stakeholders is harnessed to create a model of the system they inhabit. In this paper we focus on this aspect of fuzzy cognitive mapping, however the results are applicable to other uses of FCMs as well.

Within the social sciences participatory fuzzy cognitive mapping is used across a wide range of disciplines, such as environmental management, organisational management, urban design and industrial ecology [3,5,6,9,11,20,23,28,29]. For instance, it has recently been used to better understand such disperse systems as land cover in the Brazilian Amazon [24], the future of water in the Seyhan Basin [3], and as a method for using stakeholders in the product development process [11]. This methodology enables the creation of semi-quantitative models representing the knowledge of 'on-the-ground' experts in the system. These models can then be analysed by the researcher to gain insight into the beliefs inherent in the system, and by the participants to develop their

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own understanding of the system and how all its constituent parts interact.

Participatory fuzzy cognitive mapping generally consists of a group exercise in which the stakeholders collaborate to create a semi-quantitative model of the system. This model takes the form of a list of key concepts and a list of weighted causal links between these concepts. First of all the key concepts are discussed and agreed upon. Then causal links are discussed, along the lines of: if concept x increased or decreased would that have a *direct* effect on concept y. This creates a directed network (known as a cognitive map [15]). The final stage of the initial modelling is to assign 'fuzzy' weights, that is values from a closed set of weights, to the edges of the directed graph. The specific way that this is done varies from case to case. For example, it could be done by (i) assessing whether the link is positive (an increase in x causes an increase in y) or negative (an increase in x causes a decrease in y), or (ii) by assessing whether the link is positive or negative and whether it is weak, medium or strong [1,23,24]. As an example of this participatory methodology in practice, in a study of bio-based energy production in the Humber region [23] participants identified 16 key concepts with 27 links between them, the participants then assessed the direction and strength of all of these links. For instance, it was reasoned that any increase in bio-based energy production would lead to an increase in the number of jobs and that this causal effect was of medium strength (the weights were determined by ranking the importance of the links). Thus, that edge was labelled positive-medium.

The qualitative weights are assigned numerical values to form a quantitative model (the fuzzy cognitive map (FCM)). Several different choices are made in the literature for the assigning of such numerical values, typically lying in the interval $[-1, 1]$ [1,13,23,24], for instance [24] uses weights in the set $[-0.7, -0.5, -0.2, 0, 0.2, 0.5, 0.7]$ for links which are negative or positive and strong, medium or weak.

The process of creating a model with stakeholders is often worthwhile in and of itself for its engagement value, and for the insights offered to the researcher by stakeholders knowledgeable in the 'real' behaviour of the system. However, the use of FCMs does not stop there, the weighted network of causal relations is often turned into an iterative map [9,14,20,21,24] which is used to update values assigned to the key concepts. This requires some initial estimate of the values of the concepts. As the concepts are often all widely different and may be subjective, it is hard to assign them values, however this issue can be resolved by normalising to some 'best' case and some 'worst' case value, meaning that the values of the concepts are bounded, typically to the interval $[-1, 1]$ or $[0, 1]$. The iterative process is given by the mapping

$$\underline{x}_{n+1} = f(A\underline{x}_n), \tag{1}$$

where $\underline{x}_n$ is a vector consisting of the values of the concepts at 'time' n, and A is the directed, weighted adjacency matrix. Various monotonic functions f have been taken in the literature, such as step functions, sigmoidal functions, ramp functions and linear functions [9,14,20,21,24], sometimes with slightly different implementation procedures.

In [9] it is suggested that in analysing the map (1) one should focus on the fixed points, as for long term policy decisions the initial transient dynamics are not of interest. Furthermore since 'time' is not defined in the modelling process it is very hard to translate and interpret temporal dynamics. Focussing on fixed points means that one of the most important questions concerns the uniqueness of stable fixed points. This question shall form the main focus of this paper, and in particular we shall find conditions which guarantee the uniqueness of a fixed point for both linear and sigmoidal FCMs. This is particularly important in such a 'fuzzy' participatory setting and often appears to be implicitly assumed. If there were two stable fixed points then a slight change in initial conditions (which cannot be quantified exactly) may result in a totally different outcome, making the value of the map hard to justify. Some preliminary work has already been done seeking to address this question. For instance [9], citing [19], shows that if f is a three region piecewise linear ramp function (between 0 and 1) then (1) has a unique fixed point. Another example of work in this area is [17, Theorem 4] which gives a tight bound on when a sigmoidal FCM has a unique fixed point for a given adjacency matrix.

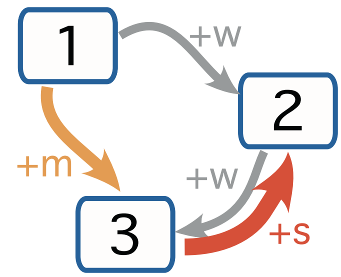

Fig. 1. A simple qualitatively weighted directed network.

Thus it seems sensible to study the existence and stability of solutions to (1) in some generality. This is what we do in this paper. Primarily we focus on general linear and sigmoidal maps and prove results about the existence, uniqueness and stability of fixed points in such systems. For instance we present a similar result to that in [17] for sigmoid FCMs, we give a generic weak bound when the adjacency matrix is not known a priori (when creating an FCM it would be beneficial to know beforehand that there will be a unique fixed point). The proof of this result is simpler than that of the similar result in [17].

It also appears that little thought has been given to the way in which the function used affects the behaviour of the iterative map, and in particular how it affects the existence and stability of fixed points, and the ordering of the concepts there. Exceptions are [27] which compares the output of three different types of FCM and [2] which compares the output of four different types of FCM. So before concentrating on the existence and stability of fixed points we first present a simple example which shows how drastically the function used, f, can affect the ordering of the concepts at the fixed point.

1.1. Example

We present a simple example with three concepts (see Fig. 1) that highlights some of the difficulties in implementing and interpreting FCMs with different functional forms f. We shall use three functional forms for f: linear, sigmoidal and step.

We use the weights 0.3 for a weak link, 0.5 for a medium strength link and 0.8 for a strong causal link. This means that we can express all the information contained in the network via (the transpose of) its adjacency matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0.3 & 0 & 0.8 \\ 0.5 & 0.3 & 0 \end{pmatrix}.$$

With this matrix we can iterate the map (1) with different functions f. However as concept 1 has no input its value will immediately take on the value $f(0)$ and remain at that value thereafter. If this value is zero (it typically is in the linear case) then this will cause all concept values to tend to zero. To avoid this scenario such a concept is generally made a 'driver' of the system [14,24] and given a self reinforcing loop of weight 1 (generally, a concept which has a self reinforcing loop is referred to as a driver). We do this here,