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An annotated bibliography on 1-planarity

Stephen G. Kobourov [a], Giuseppe Liotta [b], Fabrizio Montecchiani [b,*]

[a] Department of Computer Science, University of Arizona, Tucson, USA
[b] Dipartimento di Ingegneria, Università degli Studi di Perugia, Perugia, Italy

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ABSTRACT

The notion of 1-planarity is among the most natural and most studied generalizations of graph planarity. A graph is 1-planar if it has an embedding where each edge is crossed by at most another edge. The study of 1-planar graphs dates back to more than fifty years ago and, recently, it has driven increasing attention in the areas of graph theory, graph algorithms, graph drawing, and computational geometry. This annotated bibliography aims to provide a guiding reference to researchers who want to have an overview of the large body of literature about 1-planar graphs. It reviews the current literature covering various research streams about 1-planarity, such as characterization and recognition, combinatorial properties, and geometric representations. As an additional contribution, we offer a list of open problems on 1-planar graphs.

Contents

* Corresponding author.
E-mail addresses: kobourov@cs.arizona.edu (S.G. Kobourov), giuseppe.liotta@unipg.it (G. Liotta), fabrizio.montecchiani@unipg.it (F. Montecchiani).

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1. Introduction

Relational data sets, containing a set of objects and relations between them, are commonly modeled by graphs, with the objects as the vertices and the relations as the edges. A great deal is known about the structure and properties of special types of graphs, in particular *planar graphs*. The class of planar graphs is fundamental for both Graph Theory and Graph Algorithms, and is extensively studied. Many structural properties of planar graphs are known and these properties can be used in the development of efficient algorithms for planar graphs, even where the more general problem is NP-hard [1].

Most real world graphs, however, are *non-planar*. For example, scale-free networks (which can be used to model web-graphs, social networks and biological networks) consist of sparse non-planar graphs. To analyze such real world networks, we need to address fundamental mathematical and algorithmic challenges for *sparse non-planar* graphs, which we call *beyond-planar graphs*. Beyond-planar graphs are more formally defined as non-planar graphs with topological constraints, such as forbidden crossing patterns, as is the case with graphs where the number of crossings per edge or the number of mutually crossing edges is bounded by a constant (see, e.g., [2–4]).

A natural motivation for studying beyond-planar graphs is visualization. The goal of graph visualization is to create a good geometric representation of a given abstract graph, by placing all the vertices and routing all the edges, so that the resulting drawing represents the graph well. Good graph drawings are easy to read, understand, and remember; poor drawings can hide important information, and thus may mislead. Experimental research has established that good visualizations have a number of geometric properties, known as *aesthetic criteria*, such as few edge crossings, symmetry, edges with low curve complexity, and large crossing angles; see [5–8].

Visualization of large and complex networks is needed in many applications such as biology, social science, and software engineering. A good visualization reveals the hidden structure of a network, highlights patterns and trends, makes it easy to see outliers. Thus a good visualization leads to new insights, findings and predictions. However, visualizing large and complex real-world networks is challenging, because most existing graph drawing algorithms mainly focus on planar graphs, and consequently have made little impact on visualization of real-world complex networks, which are non-planar. Therefore, effective drawing algorithms for beyond-planar graphs are in high demand from industry and other application domains.

The notion of 1-planarity is among the most natural and most studied generalizations of planarity. A graph is 1-planar if there exists an embedding in which every edge is crossed at most once. The family of 1-planar graphs was introduced by Ringel [9] in 1965 in the context of the simultaneous vertex–face coloring of planar graphs. Specifically, if we consider a given planar graph and its dual graph and add edges between the primal vertices and the dual vertices, we obtain a 1-planar graph. Ringel was interested in a generalization of the 4-color theorem for planar graphs, he proved that every 1-planar graph has chromatic number at most 7, and conjectured that this bound could be lowered to 6 [9]. Borodin [10] settled the conjecture in the affirmative, proving that the chromatic number of each 1-planar graph is at most 6 (the bound is tight as for example the complete graph K_6 is 1-planar and requires six colors).

The aim of this paper is to present an annotated bibliography of papers devoted to the study of combinatorial properties, geometric properties, and algorithms for 1-planar graphs. Similar to [11], this annotated bibliography reports the references in the main body of the sections; we believe that this choice can better guide the reader through the different results, while reading the different sections. All references are also collected at the end of the paper, in order to give the reader an easy access to a complete bibliography.

Paper organization. The remainder of this paper is structured as follows.

- Section 2 contains basic terminology, notation, and definitions. It is divided in two main parts as follows. Section 2.1 introduces basic definitions and notation about graphs, drawings, and embeddings; this section can be skipped by readers familiar with graph theoretic concepts and graph drawing. Section 2.2 contains more specific definitions for 1-planar graphs and for subclasses of 1-planar graphs such as IC-planar and NIC-planar graphs. Because different papers on 1-planarity often use different terminology and notation, this section is important for the rest of the paper.
- Section 3 is concerned with two fundamental and closely related aspects of 1-planar graphs. Specifically, Sections 3.1 and 3.2 survey known results for the problem of characterizing and recognizing 1-planar graphs, respectively. While the recognition problem is NP-complete for general 1-planar graphs, interesting characterizations and polynomial-time recognition algorithms are known for some meaningful classes of 1-planar graphs, such as optimal 1-planar graphs.
- Section 4 investigates structural properties of 1-planar graphs, i.e., properties that depend only on the abstract structure of these graphs. Specifically, this section contains: the main results on the chromatic number, the chromatic index, and on other coloring parameters (Section 4.1); upper and lower bounds on the edge density for several classes