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A ground state solution for an asymptotically periodic quasilinear Schrödinger equation[☆]

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ABSTRACT

This article is concerned with the existence of positive ground state solutions for an asymptotically periodic quasilinear Schrödinger equation. By using a change of variables, the quasilinear problem is transformed into a semilinear one. Then, we use a Nehari-type constraint to get the existence result.

1. Introduction and main result

In this paper, we are concerned with the existence of ground state solution for the following quasilinear Schrödinger equation:

$$-\Delta u + V(x)u - (\Delta|u|^2)u = g(x, u), \quad x \in \mathbb{R}^N \tag{1.1}$$

where $N \geq 3$, V, g are asymptotically periodic functions in x. Solutions of problem (1.1) are related to the standing wave solutions for quasilinear Schrödinger equations of the form

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + V(x)\psi - g(x, |\psi|^2)\psi - \kappa[\Delta\rho(|\psi|^2)]\rho'(|\psi|^2)\psi,$$

which have been derived as models of several physical phenomena corresponding to various types of ρ. It appears in the theory of Heisenberg ferromagnets and magnons [1–5], in condensed matter theory [6], in dissipative quantum mechanics [7] and in plasma physics and fluid mechanics [8–12]. The case of $\rho(s) = (1 + s)^{1/2}$ models the self-channeling of a high-power ultra short laser in matter, see [13–16]. Problem (1.1) is the case of $\rho(s) = s$, which has been called the superfluid film equation in plasma physics, see [8,9]. Recently, problem (1.1) has been studied by several authors, see [17–29] and the references therein. The main mathematical difficulty with problem (1.1) is caused by the nonlinearity involving second order derivatives, the natural functional corresponding to problem (1.1) is not well defined for all $u \in H^1(\mathbb{R}^N)$ if $N \geq 2$. To overcome this difficulty, various arguments have been developed. In [25] and [26], by using a constrained minimization argument, a positive ground state solution has been proved for problem (1.1) with $g(x, u) = \lambda|u|^{q-1}u$, $4 \leq q + 1 < 22^*$, where $2^* = 2N/(N - 2)$ is the Sobolev critical exponent. In [22], by utilizing the Nehari method, Liu et al. obtained positive

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and sign-changing solutions. Then, by a change of variables, the quasilinear problem is transformed into a semilinear one, see [27] for an Orlicz space framework and [21] for a Sobolev space frame. Recently, a perturbation method was developed in [29] to deal with problem (1.1) which can be applied to more general quasilinear Schrödinger equation.

The study of solutions for asymptotically periodic semilinear Schrödinger equation has made great progress and attracted many authors' attention, see [30–33] and their references. We would like to point out that in a recent paper [30], Liu et al. have given reformative conditions which unify the asymptotic processes of $V,\ g$ at infinity, one can see the definitions of $\mathscr{F}_0$ and $\mathscr{F}$ below for detail. To the best of our knowledge, there is no work concerning with such conditions on V and g for the quasilinear Schrödinger equation. So, we borrow an idea from [30] to obtain the ground state solution for problem (1.1). In [34–36], they consider the quasilinear asymptotically periodic Schrödinger equation with subcritical or critical growth. However, by using the Mountain Pass Theorem, they obtained only the existence of nontrivial solutions. Here, we consider the ground state solution for problem (1.1), which has great physical interests.

There are several difficulties in our paper. The main one is the reformative condition which unifies the asymptotic processes of $V,\ g$ at infinity. This is what makes the present problem more complicated. The main argument of the proof consists rather careful estimates between g_0 and g, V_0 and V (see Lemmas 3.5 and 3.6 in Section 3 for detail) which are much more precise than the ones seen so far. Besides, we are working on $\mathbb{R}^N$ suggests that we may have to face a lack of compactness. We try to show some ways to overcome this obstacle. At last, the nonlinear term g in our paper need not be differentiable, then the constrained manifold need not be of class C^1 in our case. To get over this difficulty, we employ a similar argument in [30].

We suppose that V satisfies the following assumption:

(V) $0 < V_{\min} \le V(x) \le V_0(x) \in L^\infty(\mathbb{R}^N)$ and $V(x) - V_0(x) \in \mathscr{F}_0$,

$$\mathscr{F}_0 := \{k(x) : \forall \epsilon > 0,\ \lim_{|y|\to\infty} \text{meas}\{x \in B_1(y) : |k(x)| \ge \epsilon\} = 0\},$$

and V_0 satisfies $V_0(x+z) = V_0(x)$ for all $x \in \mathbb{R}^N$ and $z \in \mathbb{Z}^N$.

Setting $G(x,s) := \int_0^s g(x,t)dt$, the function $g \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R})$ satisfies

(g_1) $\lim_{s\to 0^+} \frac{g(x,s)}{s} = 0$ uniformly for $x \in \mathbb{R}^N$.

(g_2) $\lim_{s\to\infty} \frac{g(x,s)}{s^{2\cdot 2^*-1}} = 0$ uniformly for $x \in \mathbb{R}^N$.

(g_3) $s \mapsto \frac{g(x,s)}{s^3}$ is nondecreasing on $(0,+\infty)$.

(g_4) there exists $g_0 \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R}^+)$ such that

(1) $g(x,s) \ge g_0(x,s)$ for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}^+$ and $g(x,s) - g_0(x,s) \in \mathscr{F}$, where $\mathscr{F} := \{h(x,s) : \forall \epsilon > 0,\ \lim_{|y|\to\infty} \text{meas}\{x \in B_1(y) : |h(x,s)| \ge \epsilon\} = 0$ uniformly for $|s|$ bounded$\}$.

(2) $g_0(x+z,s) = g_0(x,s)$ for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}^+$ and $z \in \mathbb{Z}^N$.

(3) $s \mapsto \frac{g_0(x,s)}{s^3}$ is nondecreasing on $(0,+\infty)$.

(4) $\lim_{s\to\infty} \frac{G_0(x,s)}{s^4} = +\infty$ uniformly for $x \in \mathbb{R}^N$.

Now, we state our main result:

Theorem 1.1. *Suppose that (V) and (g_1)–(g_4) are satisfied, then problem (1.1) possesses a positive ground state solution.*

In the particular case: $V = V_0, g = g_0$, we can get a solution for the periodic problem from Theorem 1.1. That is, considering the problem

$$-\Delta u + V_0(x)u - (\Delta|u|^2)u = g_0(x,u),\quad x \in \mathbb{R}^N \tag{1.2}$$

under the hypothesis:

(V_0) the function $V_0(x)$ satisfies $0 < \inf_{x\in\mathbb{R}^N} V_0(x) \le V_0(x) \in L^\infty(\mathbb{R}^N)$ and $V_0(x+z) = V_0(x)$ for all $x \in \mathbb{R}^N$ and $z \in \mathbb{Z}^N$.

We can obtain the existence result for the periodic problem:

Corollary 1.2. *Suppose that (V_0) holds, g_0 satisfies (g_1)–(g_3), $(g_4$-$2)$ and $(g_4$-$4)$. Then,* Eq. (1.2) *possesses a positive ground state solution.*

Remark 1.3. To the best of our knowledge, even for the periodic case, our result is new. In [37], under condition (V_0), g_0 satisfies (g_1), (g_3), $(g_4$-$2)$, $(g_4$-$4)$ and

(g_2') $|g_0(x,u)| \le a(1+|u|^{p-1})$ for some $a > 0$ and $4 < p < 22^*$.

They could get a positive solution for problem (1.2). Our condition (g_2) is somewhat weaker than (g_2').

Notation: In this paper, we use the following notations:

- $H^1(\mathbb{R}^N)$ is the usual Hilbert space endowed with the norm

$$\|u\|_H^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right)dx.$$