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Discontinuous Galerkin methods with staggered hybridization for linear elastodynamics

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ABSTRACT

In this paper, we will develop a new staggered hybridization technique for discontinuous Galerkin methods to discretize linear elastodynamic equations. The idea of hybridization is used extensively in many discontinuous Galerkin methods, but the idea of staggered hybridization is new. Our new approach offers several advantages, namely energy conservation, high-order optimal convergence, preservation of symmetry for the stress tensor, block diagonal mass matrices as well as low dispersion error. The key idea is to use two staggered hybrid variables to enforce the continuity of the velocity and the continuity of the normal component of the stress tensor on a staggered mesh. We prove the stability and the convergence of the proposed scheme in both the semi-discrete and the fully-discrete settings. Numerical results confirm the optimal rate of convergence and show that the method has a superconvergent property for dispersion. Furthermore, an application of this method to Rayleigh wave propagation is presented.

1. Introduction

Accurate elastic wave simulations are of critical importance in a variety of geophysical applications. One popular class of methods is the staggered grid finite difference methods proposed by [1–3], which are very efficient for regular domains with flat interfaces or surfaces. However, it is not easy to apply these methods to domains with complex geometries or nonflat interfaces, which arise in more realistic applications. Recently, the discontinuous Galerkin (DG) method has become increasingly popular, due to its great flexibility for higher-order spatial approximations and its ability of computing on irregular domains. For instances, [4] demonstrated the *hp*-adaptivity for 3-D elastic wave with convolution perfectly matched layer, [5] developed *hp*-adaptivity schemes for elastic scattering, [6] designed a numerical scheme with efficient parallel implementation for 3-D wave propagation problem in coupled elastic–acoustic media, and [7] proposed a solution strategy for this problem using *hp*-adaptivity.

For a linear elastic material, wave propagation in the domain $\Omega \subset \mathbb{R}^d$ is governed by

$$\rho \frac{\partial^2 \mathbf{w}}{\partial t^2} - \operatorname{div} \mathbf{C}\boldsymbol{\varepsilon}(\mathbf{w}) = \mathbf{f} \quad \text{in } \Omega, \tag{1}$$

where $d = 2$ or 3, $\rho > 0$ is the mass density, $\mathbf{w}(t, \mathbf{x}) : [0, \infty) \times \Omega \to \mathbb{R}^d$ is the displacement, $\mathbf{C}$ is the stiffness tensor of the medium, $\boldsymbol{\varepsilon}(\mathbf{w}) = \frac{1}{2}(\nabla \mathbf{w} + \nabla \mathbf{w}^T)$ is the strain tensor and $\mathbf{f}$ is the external force. There are two commonly used formulations for the discretization. The *displacement–stress formulation* solves this problem in terms of the displacement and/or stress.

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Examples of DG method of this type, include the Internal Penalty Discontinuous Galerkin (IPDG) methods in [8–10]. In addition, [11] proposed a DG method of this type where the energy is conserved with a suitable choice of numerical fluxes.

On the other hand, the *velocity–stress formulation* or the *velocity–strain formulation* solves this problem in terms of the velocity and the stress or strain, resulting in a first-order system. The displacement can be recovered by integrating the velocity in time. A variety of semi-discrete DG methods have been proposed to solve this system, e.g. [4,6,12,13]. These methods use discontinuous elements to discretize the space with time-stepping to solve the system. To achieve higher accuracy in time, [14] introduced the ADER-DG method for two-dimensional isotropic elastic wave propagation, where the upwind flux and the ADER scheme in time are used to get the same order of accuracy in space and in time.

Recently, a new class of staggered DG (SDG) methods based on staggered meshes has been successfully developed for the first-order formulation of wave equations. In [15,16], the SDG method was proposed for the time-dependent acoustic wave equation. Note that the idea for the acoustic-wave equation cannot be directly applied to the elastic-wave equation due to the symmetry of the stress tensor. In [17], a new SDG method using the Lagrange multiplier technique for the enforcement of the symmetry of the stress tensor was constructed. However, due to the staggered continuity requirements on basis functions, this SDG method only gives a weak symmetry condition for the stress tensor.

In this paper, we introduce a new idea for elastic wave simulations. In particular, we will develop a new staggered hybridization technique. We remark that the idea of hybridization is used extensively in many discontinuous Galerkin methods, such as [13,18–22], but the idea of staggered hybridization is new. In our new method, the construction of a staggered mesh follows the ideas from [15,16,23], where SDG methods were proposed for the time-dependent acoustic wave equation and the static linear elastic equation. The new method is based on piecewise polynomial approximations. Compared to [17], the continuity requirements on basis functions are removed, and the symmetry of the stress tensor is strongly enforced in the approximation space. To couple the basis functions across element boundaries, two staggered hybrid variables are used. These two hybrid variables are defined on edges of the staggered mesh, and are used to enforce the continuity of the velocity and the continuity of the normal component of the stress tensor in a staggered way. The resulting scheme offers several advantages, namely energy conservation, high-order optimal convergence as well as low dispersion error. Moreover, the new scheme preserves the strong symmetry for the stress tensor. With respect to the time-stepping, the new scheme requires only solutions of local saddle point problems defined on unions of few elements, and is thus very efficient. This "local" feature is the result of our staggered hybridization. In addition, our method is locking-free. In other words, the convergence is independent of the first Lamé's parameter λ. For nearly incompressible materials, λ is very large and many standard methods fail to address these materials as the numerical error grows as λ increases. We remark that [17,24] show that the SDG scheme produces numerical solutions with dispersion errors that are two order higher than non-staggered DG schemes. We will show numerically that the new scheme proposed in this paper also has a high order dispersion error. We will also show an application involving the simulations of the Rayleigh waves.

The rest of this paper is organized as follows. The paper starts with the problem setting in Section 2. It is followed by the detail of the proposed method in Section 3. Next, analyses for the convergence, the conservation, and the stability of the semi-discrete solutions are given in Section 4. Analyses for fully discrete solutions are given in Section 5. Numerical examples are given in Section 6 to demonstrate the performance of the proposed method. Finally, in Section 7, we give some conclusions.

2. Problem setting

Denote the velocity $\mathbf{u} := \mathbf{w}_t$, the $d \times d$ stress tensor $\boldsymbol{\sigma} := \mathbf{C}\boldsymbol{\varepsilon}(\mathbf{w})$ and the compliance tensor $\mathbf{A} := \mathbf{C}^{-1}$. We rewrite (1) as the following first order system,

$$\rho \frac{\partial \mathbf{u}}{\partial t} - \operatorname{div} \boldsymbol{\sigma} = \mathbf{f}, \tag{2}$$

$$\mathbf{A}\frac{\partial \boldsymbol{\sigma}}{\partial t} - \boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{0}, \tag{3}$$

in a bounded domain $\Omega \subset \mathbf{R}^d$ $(d = 2, 3)$ with a Lipschitz boundary. We consider the time t that lies in the interval $[0, T]$.

To fix the notation, we write $\mathbf{u} := (u_1, \dots, u_d)^T$, $\boldsymbol{\sigma} := (\sigma_{ij})$ and $\mathbf{f} := (f_1, \dots, f_d)^T$. We let $\boldsymbol{\sigma}_i$ be the ith row of $\boldsymbol{\sigma}$ and define the divergence as $\operatorname{div}\boldsymbol{\sigma} := (\operatorname{div}\boldsymbol{\sigma}_1, \dots, \operatorname{div}\boldsymbol{\sigma}_d)^T$. Moreover, $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is a symmetric matrix, where $\nabla\mathbf{u} := \left(\partial_j u_i\right)$ is the row-wise gradient of $\mathbf{u}$.

In the rest of this paper, we assume the elastic medium is homogeneous and isotropic. In other words, the compliance tensor $\mathbf{A}$ is given by

$$\mathbf{A}\boldsymbol{\tau} := \frac{1}{2\mu}\left(\boldsymbol{\tau} - \frac{\lambda}{2\mu + d\lambda}\operatorname{tr}(\boldsymbol{\tau})\,\mathbf{I}\right), \tag{4}$$

where λ and μ are positive constants called the first and the second Lamé's parameters, respectively, $\operatorname{tr}(\boldsymbol{\tau})$ is the trace of $\boldsymbol{\tau}$ and $\mathbf{I}$ is the identity tensor.