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A low-order finite element method for three dimensional linear elasticity problems with general meshes

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The paper is concerned with a low-order finite element method, namely the staggered cell-centered finite element method, which has been proposed and analyzed in Ong et al. (2015) for two-dimensional compressible and nearly incompressible linear elasticity problems. In this work, we extend the results to the three-dimensional case and focus on the creating of the meshes. In particular, from a general primal mesh $\mathcal{M}$, we construct a polygonal dual mesh $\mathcal{M}^*$ and its submesh $\mathcal{M}^{**}$ in a way such that each dual control volume of $\mathcal{M}^*$ corresponds to a primal vertex and is a union (macro-element) of some fixed number of adjacent tetrahedral elements of $\mathcal{M}^{**}$. The displacement is approximated by piecewise trilinear functions on the subdual mesh $\mathcal{M}^{**}$ and the pressure by piecewise constant functions on the dual mesh $\mathcal{M}^*$. As for two-dimensional case, such construction of the meshes and approximation spaces satisfies the macroelement condition, which implies stability and convergence of the scheme. Numerical experiments are carried out to investigate the performance of the proposed method on various mesh types.

1. Introduction

Numerical methods for finding approximate solutions of three-dimensional (3D) boundary value problems on general meshes are necessarily required for various engineering applications. In particular, many numerical schemes have been proposed and analyzed for the 3D linear elasticity problems in order to give more accurate numerical solutions in different cases, which helps to efficiently model and simulate the corresponding physical processes. Clearly, low-order finite elements offer simpler implementation and cheaper computation cost than higher-order formulations at the same mesh size. For example, 3D low-order elements were proposed in [1,2]. However, these elements do not converge uniformly with respect to an incompressible or nearly incompressible behavior, since they have the poor performance in the numerical patch tests [3] and violate the inf–sup conditions [4,5]. On the contrary, the finite volume methods are efficiently used in structural analysis of solids, possibly in combination with finite element solvers. There are many examples of the finite volume discretization in linear stress analysis problems that can be found in the literature [6–8]. These methods are second-order accurate on control volumes, but they additionally employ multigrid solvers to speed-up convergence and they can handle only admissible meshes. A development of finite volume methods using the multi-point flux approximation (MPFA), namely the multi-point

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stress approximation (MPSA) [9] (including three types O-MPSA, U-MPSA and L-MPSA), can be applied to problems with anisotropic, heterogeneous and discontinuous coefficients. Nevertheless, these methods may be less stable and apparently non-convergent in presence of distorted meshes such as Keshaw mesh (their coercivity is not uniform with mesh) and the strong anisotropy coefficients [10,11]. Furthermore, except in some particular cases, it is claimed in [9,12,13] that these multi-point schemes usually fail to meet two important properties, namely symmetry and positive-definiteness.

With those drawbacks of low-order elements, many higher order finite elements were proposed to overcome the volumetric locking effect such as the MINI method [14] on tetrahedral meshes in which the approximate spaces for the displacement and pressure are the trilinear finite element space enriched with bubble functions and the linear finite element space respectively. Such an enrichment technique were also used in the bFS-FEM method [15], where the approximate strain and divergence are smoothed on the dual mesh. In addition, the solution of nearly incompressible linear elasticity problem can be approximated by the mixed finite element methods in which extra variables are defined by following a saddle point framework. Mixed formulations with the trilinear displacement and piecewise constant pressure are often used to overcome the volumetric locking, see [16–18]. Another approach is to use nonconforming finite elements [19,20] which have been successively extended to the 3D linear elastic problems. Their uniform convergence was analyzed in both two and three dimensions [20–22]. The mimetic finite difference (MFD) methods [23–25] can also be applied to the elasticity problems in the nearly incompressible case as well as to Stokes problems in the incompressible case on general 3D meshes. In addition, in [26] the enhanced strain methods were proposed by enriching the strains arising from the discrete displacement field by means of suitable local modes. However, their numerical results detected in [27] are instable (hourglassing) in areas of large compression. To get the hourglass stabilization, the concept of reduced integration with hourglass stabilization was suggested in [28–30] and [31] (these formulations can also be applied to non-linear elasticity or even plasticity problems). Moreover, the 3D discrete duality finite volume method [32] has been constructed for linear elasticity problems. However, it is subtle to preserve the discrete duality property, which is used to derive optimal estimates of the discrete error.

Those methods listed above are only applicable to some certain types of mesh since the construction of their basis functions depends severely on the mesh. For example, basis functions of the conforming, the nonconforming, MINI and the mixed finite element methods can be constructed only on tetrahedral or hexahedral meshes, while theoretical analysis of the MFD methods is restricted to the mesh regularity assumptions [33, p. 1173]. Only a few methods can handle general meshes, for instance, in [34] the authors have proposed the variable-element-topology finite element method (VETFEM) for 3D problems with arbitrary polyhedral elements. In addition, recently developed virtual element methods [35,36] allow high-order accuracy and high-order continuity while overcoming volumetric locking effect of 2D and 3D nearly incompressible linear elasticity problems on arbitrary meshes. Furthermore, isogeometric methods based on isogeometric analysis [37] have been proposed in [38,39] in which the authors investigate the use of higher-order Non-Uniform Rational B-Splines (NURBS) within the $\overline{B}$ projection method. The methods provide high-order optimally-accurate solutions for all Poisson ratios. Overall, almost all existing methods still have one of the following restrictions: (i) they have the poor performance because of the locking effect; (ii) they are not able to deal with general meshes, or they have difficulty in constructing dual meshes; (iii) they have quite expensive computational cost since they use either high-order finite elements or low-order elements with not only cell unknowns but also extra unknowns on the edges/faces or vertices to approximate the displacement field.

In this work, we present the 3D extension of the staggered cell-centered finite element method (SC-FEM) for compressible and nearly incompressible linear elasticity problems on general meshes. The method inherits full advantages of the two-dimensional SC-FEM [40], in particular:

(i) it uses low-order finite approximation spaces for the displacement (piecewise trilinear functions on the tetrahedral subdual mesh) and for the pressure (piecewise constant functions on the dual mesh) while still satisfying stability condition (through the use of the macroelement condition [41]);

(ii) different types of 3D meshes can be used since the scheme is defined on the dual mesh and its submesh which can be constructed from a given possibly distorted general mesh;

(iii) the scheme is efficient in terms of accuracy and solver cost since the displacement is approximated on the subdual mesh which is finer than the primal mesh and the meshes are constructed in a way to recover a cell-centered scheme, i.e. only cell-unknowns of primal elements (for the displacement) and of dual control volumes (for the pressure) are involved in the SC-FEM. Note that by construction, the scheme requires also an extra cost of the meshes generation procedure.

Additionally, giving the mathematical equivalence between classical elasticity problems and Stokes problems in fluid mechanics, the proposed cell-centered scheme can be extended to solve Stokes problems with a constant viscosity by using the stabilization technique given by the well-known penalty method [42,43] in the finite element framework. Moreover, as the SC-FEM is mostly based on the ideas of the finite element cell-centered scheme [44,45] for 2D and 3D anisotropic, heterogeneous diffusion problems, it can be extended to deformable porous media problems as in [9] or Stokes problems with piecewise Lipschitz-continuous viscosity in which local continuity of stress or flux is satisfied.

In the paper, we shall focus on the description of the construction of the meshes used in the SC-FEM, i.e. the creating of the dual mesh and its tetrahedral submesh from a given primal mesh. Theoretical results concerning the well-posedness and convergence of the discrete problems are omitted since they are just the same as in the 2D case [40]. The calculation of the associated linear algebraic systems for 3D problems is presented in detail for implementation purpose. Numerical behaviors of the proposed method for different 3D benchmark test cases with general, possibly nonconforming or distorted meshes are discussed.