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Reconstruction of space-dependent potential and/or damping coefficients in the wave equation

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ABSTRACT

In this paper, nonlinear reconstructions of the space-dependent potential and/or damping coefficients in the wave equation from Cauchy data boundary measurements of the displacement and the flux tension are investigated. This is a very interesting and challenging nonlinear inverse coefficient problem with important applications in wave propagation phenomena. The uniqueness and stability results that are revised and in some cases proved demonstrate an advancement in understanding the stability of the inverse coefficient problems. However, in practice, the inverse coefficient identification problems under investigation are still ill-posed since small random errors in the input data cause large errors in the output solution. In order to stabilize the solution we employ the nonlinear Tikhonov regularization method. Numerical reconstructions performed for the first time are presented and discussed to illustrate the accuracy and stability of the numerical solutions under finite difference mesh refinement and noise in the measured data.

1. Introduction

Many practical applications related to wind, wave, seismic or noise excitations require reconstructing the applied loadings/forces/sources from the knowledge of output responses. For example, in [1], time-dependent external forces in a nonlinear damped vibration system were retrieved from the knowledge of the displacement and velocity at different times. Another application of interest concerns distinguishing between various types of seismic events, e.g., explosion, implosion or earthquake, which generate waves that propagate through the ground and can be recorded using seismometers. In [2], a seismic source modelled as a point moment tensor in the elastic wave equation was estimated from time-dependent wave form measurements. A final related application that is mentioned concerns inverse problems in ocean acoustics, in which the point forces/sources of the ocean seafloor are determined from acoustic pressure measurements on an array of hydrophones, see [3].

The above practical applications can be viewed in a unified mathematical way as inverse force problems for the hyperbolic wave equation

$$u_{tt} - \mathcal{L}u = F(x, t, u, u_t, \nabla u),$$

where the operator $\mathcal{L} = \nabla^2$ is the Laplacian for homogeneous media, and $\mathcal{L} = c(x)\nabla^2$ or $\mathcal{L} = \nabla \cdot (\mu(x)\nabla)$ for inhomogeneous media with positive physical properties $c(x)$ or $\mu(x)$, see [4], and $u(x, t)$ and $F(x, t, u, u_t, \nabla u)$ are unknown displacement and forcing term that need to be identified from prescribed initial and Cauchy, i.e. both Dirichlet and Neumann, boundary data.

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The linear case when the force $F(x)$ depends only on the space variable x was investigated in some detail both theoretically in [5–8], and recently, numerically in [9–11]. Also, the purely nonlinear case when the force $F(u)$ depends only on the displacement u, was investigated in [12]. More recently, inverse coefficient identification problems in which the force expresses as $F(x, u, u_t, \nabla u) = Q_0(x)u + Q_1(x)u_t + Q_2(x) \cdot \nabla u$, with unknown space-dependent coefficients $Q_0(x)$, $Q_1(x)$ and/or $Q_2(x)$, have been the point of interest of some theoretical studies, see [13–15]. In these studies, the powerful technique of Carleman estimates was employed, see [6,16–19]. It is the purpose of this paper to make new mathematical and numerical contribution along the lines of these nonlinear space-dependent coefficient identification problems for the wave equation.

The plan of the paper is as follows. In Section 2, we give the general setup of the inverse coefficient identification problems (ICIPs) under investigation with particular analysis performed in Section 3. The uniqueness and conditional Lipschitz-type stability of recovering the potential coefficient $Q_0(x)$ is known to hold in certain regular spaces of functions under the assumption of a non-zero initial displacement, as reviewed in Section 3.1, but similar results for recovering the damping coefficient $Q_1(x)$ are not so well-documented. Therefore, Section 3.2 is devoted to proving these new uniqueness and conditional Lipschitz-type stability results given by Theorem 4 concerning the recovery of the space-dependent damping coefficient. The proof is based on Carleman estimates for the wave equation with forcing terms and appropriate extensions of solutions and coefficients to the negative time interval. The analysis requires non-zero initial velocity being prescribed, which may be a practical limitation but this condition is essential for the applicability of the method of Carleman estimates because we must choose/control an initial velocity whose sign is the same everywhere in the closure of the space domain. However, if we change many times the initial displacement or velocity so as the union of their supports covers the closure of the space domain, then the set of all corresponding observation data can yield the same uniqueness and stability results of Theorems 1–4.

After theoretical analysis, Sections 4 and 5 describe the numerical methods used for solving the direct and inverse problems based on the finite difference discretization and nonlinear constrained minimization using the MATLAB toolbox routine *lsqnonlin*. Section 6 presents and discusses numerical results for the three ICIPs that are investigated. Various features of the investigation include the case of partial Cauchy data, inversion of data contaminated with noise and regularization. Finally, Section 7 presents the conclusions of the study and directions for possible future work.

2. Mathematical formulation

Consider a medium occupying a bounded region Ω in $\mathbb{R}^n$, with a sufficiently smooth boundary $\partial\Omega$, e.g. of class C^2. Throughout this paper, we assume that $n = 1, 2, 3$. For the case of higher dimensions, $n > 3$, we can argue similarly but we have to assume more regularity of solutions, and we do not discuss it here. Define the space–time cylinder $Q_T = \Omega \times (0, T)$, where $T > 0$. We wish to find the displacement $u(x, t)$ and the spacewise dependent coefficients $Q_0(x)$ and/or $Q_1(x)$ of the lower-order terms in the hyperbolic wave equation

$$u_{tt} = \nabla^2 u + Q_0(x)u + Q_1(x)u_t \quad \text{in } Q_T. \tag{1}$$

In Eq. (1), Q_0 is called the potential coefficient, whilst Q_1 is called the damping coefficient. In principle, we could add the extra term $Q_2(x) \cdot \nabla u$ with known or unknown vector coefficient $Q_2(x)$ to the right-hand side of (1), see [13,14], but this additional extension will be investigated in a separate work.

The initial conditions are

$$u(x, 0) = \varphi(x), \quad x \in \Omega, \tag{2}$$

$$u_t(x, 0) = \psi(x), \quad x \in \Omega, \tag{3}$$

where $\varphi(x)$ and $\psi(x)$ represent the initial displacement and velocity, respectively. On the boundary we can prescribe Dirichlet, Neumann, Robin or mixed boundary conditions.

Let us consider, Neumann boundary conditions being prescribed, namely,

$$\frac{\partial u}{\partial \nu}(x, t) = q(x, t), \quad (x, t) \in \partial\Omega \times (0, T), \tag{4}$$

where q is a given function.

If the functions Q_0 and Q_1 are given, then Eqs. (1)–(4) form a direct well-posed problem. However, if some of the functions Q_0 and/or Q_1 cannot be directly observed they hence become unknown and then clearly, the above set of equations is not sufficient to determine uniquely the solution of the so-generated ICIP. In order to compensate for this non-uniqueness, we consider the additional measurement given by the Dirichlet boundary data,

$$u(x, t) = P(x, t), \quad (x, t) \in \partial\Omega \times (0, T), \tag{5}$$

where P is a prescribed boundary displacement. We can also consider the case when the boundary displacement Dirichlet data (5) is being prescribed and it is the flux tension Neumann data (4) which is being measured.

Note that the unknowns $Q_0(x)$ and $Q_1(x)$ are interior quantities depending on the space variable $x \in \Omega \subset \mathbb{R}^n$, whilst the additional measurement (5) is a boundary quantity depending on $(x, t) \in \partial\Omega \times (0, T)$.