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A high-order finite difference method for option valuation

Mehzabeen Jumanah Dilloo, Désiré Yannick Tangman *

Department of Mathematics, University of Mauritius, Réduit, Mauritius

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ABSTRACT

In this paper, we propose the use of an efficient high-order finite difference algorithm to price options under several pricing models including the Black–Scholes model, the Merton's jump–diffusion model, the Heston's stochastic volatility model and the nonlinear transaction costs or illiquidity models. We apply a local mesh refinement strategy at the points of singularity usually found in the payoff of most financial derivatives to improve the accuracy and restore the rate of convergence of a non-uniform high-order five-point stencil finite difference scheme. For linear models, the time-stepping is dealt with by using an exponential time integration scheme with Carathéodory–Fejér approximations to efficiently evaluate the product of a matrix exponential with a vector of option prices. Nonlinear Black–Scholes equations are solved using an efficient iterative scheme coupled with a Richardson extrapolation. Our numerical experiments clearly demonstrate the high-order accuracy of the proposed finite difference method, making the latter a method of choice for solving both linear and nonlinear partial differential equations in financial engineering problems.

1. Introduction

In financial engineering, several methods have been proposed in literature to improve the accuracy of numerical option pricing algorithms. Low-order finite difference schemes for solving partial differential equations (PDEs) have been enhanced through the common use of extrapolation techniques [1]. Unfortunately, extrapolation is highly dependent on the smooth convergence of the numerical methods underhand and can give inaccurate results if the convergence rate is too erratic. In [2–4], the authors have used spectral methods to improve the accuracy of European style options under one-dimensional models. However, these methods require the solutions of dense linear systems in $\mathcal{R}^{N\times N}$ which have $\mathcal{O}(N^3)$ complexity, where N represents the number of nodes used to discretise the spatial domain. Although such schemes give accurate results for European options, they are less efficient for American options because spectral convergence is usually not achieved due to the difficulty in accurately capturing the free boundary location [5]. On the other hand, [6–8] have developed high-order compact (HOC) schemes which have linear complexity when computing the option prices. However, the derivation of these schemes involves the differentiation and intricate manipulations of the PDE at hand in order to replace the higher order derivatives in the leading truncation error terms [6]. Such an approach will be generally more difficult to implement for PDEs with variable and time dependent coefficients such as those that arise in the nonlinear Black–Scholes equations. Moreover, it is well known that the point of singularity that exists at the strike price hampers the order of convergence of these HOC discretisations. So far, one of the best remedial approaches was suggested in [7] where a local mesh refinement strategy is applied by adding four nodes in the vicinity of that point of singularity at each refinement stage.

* Corresponding author.
E-mail address: y.tangman@uom.ac.mu (D.Y. Tangman).

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When the assumptions of no transaction costs and a perfectly liquid market are relaxed, the Black–Scholes pricing PDE becomes nonlinear. Such problems do not admit closed-form solutions even for vanilla options, and since the nonlinear term is found in the diffusion coefficient, the problem becomes stiff. Numerically, this requires stability restrictions to be imposed on the time step if an explicit scheme is used to discretise this nonlinear term [9]. In [10], the second-order derivative within the nonlinear coefficient is approximated over twice the spatial mesh size used to discretise the derivatives with linear coefficients. Although this strategy helps to improve the stability of the implicit–explicit type of schemes, it does not completely alleviate the problem. More recently, splitting methods based on locally one-dimensional (LOD) backward Euler (implicit) method [11] and on LOD Crank–Nicolson method [12] have been suggested to solve the nonlinear Black–Scholes equations. Even though explicit formulae could be obtain while implementing the stable first-order implicit and second-order Crank–Nicolson schemes, restrictive positivity-preserving conditions on the time step are required to obtain convergent solutions to the exact solution. We also want to mention that [13,14] have proposed high-order schemes for the numerical solution of the nonlinear Black–Scholes equation, but since their methods did not cater for the singularities in the payoff functions, only a lower order convergence could be observed.

Stochastic volatility models have been developed in order to produce option prices that better describe the volatility smiles and skews of real market data, and one of the most popular stochastic volatility model is the Heston's model [15]. Several numerical methods, namely finite element-finite volume methods [16,17], spectral element approximations [18] and method of line approaches [19] among others, have been proposed to price options under this model. The resulting two-dimensionality of the problem and the presence of second-order mixed spatial partial derivatives complicate the application of high-order schemes in the pricing of financial derivatives. Indeed, in [20], the derivation of a HOC scheme for the Heston's model imposes the mesh size along each spatial direction to be equal, which can be quite restrictive. More recently, [5,21] have used the grid transformation of [22] which concentrates grid nodes at the strike price to build high-order schemes for stochastic volatility models. But this requires the evaluation of the Jacobian and the Hessian of transformations which are not very practical for multiple points of singularity.

In this paper, we propose to use the non-uniform discretisation due to Bowen and Smith [23] for option pricing since such discretisation confers the following advantages over classical spatial discretisation as it:

- grants us with added flexibility to construct our grid such that refinement points can be easily added in the vicinity of several singular points,
- helps to reduce the number of computational nodes used, whereby far-field boundary conditions [24] can be naturally implemented using a coarse grid to extend our computational domain without the use of complex artificial boundary conditions [25–27], and this significantly improves the efficiency and practicality of the PDE approach,
- results in banded pentadiagonal linear systems which have the same linear computational complexity as for tridiagonal linear systems,
- gives high-order convergent prices for European options, path dependent options such as barrier options and American options,
- gives high-order convergence for the nonlinear Black–Scholes and the Heston's two-dimensional problems,

and these of course, represent the main contributions of the present work. We also show how to solve the integro term in the Merton's jump–diffusion model in $\mathcal{O}(N)$ computations per time step improving on the $\mathcal{O}(N \log N)$ in [7,28]. Recently, [29] have used a second-order finite difference discretisation in space with a discontinuous Galerkin finite element in time to solve Merton's partial integro-differential equation. However, the multigrid algorithm used to solve the dense linear systems is less efficient compared to our proposed scheme which solves only sparse linear systems. Our non-uniform pentadiagonal scheme can be easily combined with an exact in time exponential time integration scheme [2,3,30,31] to generate highly accurate option prices. An efficient iterative scheme [32] coupled with a repeated extrapolation technique [33] is further proposed to solve the nonlinear problems.

The paper is structured as follows. In Section 2, we describe the different option pricing problems and models that we later consider in our numerical experiments. In Section 3, we give the high-order non-uniform finite difference approximations used in space and we explain the local mesh refinement strategy used to restore the high-order convergence rate. We then describe the time-stepping schemes for both linear and nonlinear problems, and we briefly discuss the stability analysis of the proposed scheme. Section 4 numerically demonstrates the efficiency and accuracy of the new scheme for option pricing, and finally, we conclude in Section 5 where we also give some promising scope for future work.

2. Option pricing

We state here the different pricing problems and models that we shall study in this paper. Under the classical Black–Scholes model, we aim at pricing various types of options which have payoffs involving different points of singularity. This is to demonstrate the versatility of our approach. Then, we consider various types of nonlinear Black–Scholes PDEs arising from popular transaction costs and illiquidity models. Further, we show how pricing can be extended to other realistic models like the Merton's jump–diffusion and Heston's stochastic volatility models.