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Existence and multiplicity of positive solutions for elliptic equation with critical weighted Hardy–Sobolev exponents and boundary singularities ☆

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ABSTRACT

We study a semilinear elliptic equation involving critical weighted Hardy–Sobolev exponents with boundary singularities. The existence and multiplicity of positive solutions are established. Our method relies upon Ekeland's variational principle and Mountain Pass Lemma.

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1. Introduction and main results

In this paper, we consider the following semilinear elliptic equation

$$\begin{cases} -\mathrm{div}(|x|^{-2a}\nabla u) - \mu \dfrac{u}{|x|^{2(1+a)}} = \dfrac{|u|^{p-2}u}{|x|^{bp}} + \lambda f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with C^2 boundary $\partial\Omega$ and $0 \in \partial\Omega$, $0 \leq a < \sqrt{\bar{\mu}}$, $\bar{\mu} \triangleq \frac{(N-2)^2}{4}$, $0 \leq \mu < (\sqrt{\bar{\mu}} - a)^2$, $a \leq b < a+1$, $p = p(a,b) \triangleq \frac{2N}{N-2(1+a-b)}$ is the critical weighted Hardy–Sobolev exponent and $2^* \triangleq p(a,a) = \frac{2N}{N-2}$ is the critical Sobolev exponent, $\lambda > 0$ is a real parameter and $f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$. Since we consider the existence of positive solutions of the problem (1), it is obvious that the values of $f(x,t)$ for $t < 0$ are irrelevant, so we may define $f(x,t) = 0$ for $x \in \Omega$, $t \leq 0$.

The problem (1) has the following form

$$-\,\mathrm{div}\,(A(x)\nabla u) = g(x,u), \tag{2}$$

where $A(x)$ is a nonnegative function which may be unbounded at some points. It is well known that Eq. (2) arises from the consideration of standing waves in the anisotropic Schrödinger equation. If $a = b = \mu = 0$ and $f(x,u) = u$, Eq. (1) reduces

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to

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

which has been studied in the celebrated paper [1].

In recent years, much attention has been paid to the singular elliptic problems with critical weighted Hardy–Sobolev exponents (the case that $a \neq 0, b \neq 0$) or critical Hardy–Sobolev exponents (the case that $a = 0, b \neq 0$) (see [2–10] and so on), mainly when $0 \in \Omega$. For example, in [4], Song and Tang considered the multiple of positive solutions for Robin problem involving critical weighted Hardy–Sobolev exponents with boundary singularities. In [6], Nyamoradi studied the multiplicity of positive solutions to some weighted nonlinear elliptic system involving critical exponents. For the boundary singularities problems ($0 \in \partial\Omega$), which have been studied by a lot of researchers yet (see [11–14] and the references therein). In particular, let us recall that, in [11], Shang considered the following semilinear elliptic problem:

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = \dfrac{|u|^{2^*(s)-2}}{|x|^s}u + \lambda f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{4}$$

By Ekeland's variational principle and Mountain Pass Lemma, he proved existence of two positive solutions when λ is less than some constant for problem (4). In [15], Ghoussoub and Kang proved the existence of the best Hardy–Sobolev constant in $H_0^1(\Omega)$ when $N \geq 4$ and the principle curvatures of $\partial\Omega$ are negative. Besides, if $f(x,t) = t$ in problem (4), they also obtained a weak solution under the assumptions $N \geq 4, \mu = 0, 0 < \lambda < \lambda_1$ (the first eigenvalue of operator $-\Delta$ on $H_0^1(\Omega)$) and the principle curvatures are non-positive. In [12], Chern and Lin proved the existence of the best weighted Hardy–Sobolev constant with the singularity on the boundary. As far as we know, the Dirichlet problem with critical weighted Hardy–Sobolev exponents and boundary singularities has not yet been studied.

In the present paper, motivated by [11,15], we do some studies for such problem. Firstly, by Ekeland's variational principle, we establish the existence of a positive local minimum for the associated functional. Due to the lack of compactness of the embedding in $H_0^1(\Omega, |x|^{-2a}) \hookrightarrow L^{2^*}(\Omega)$, the classical Palais–Smale condition in $H_0^1(\Omega, |x|^{-2a})$ fails to satisfy for the energy functional of problem (1), where $H_0^1(\Omega, |x|^{-2a})$ denotes the completion of $C_0^\infty(\Omega)$ with the standard norm $\|u\|_H = \int_\Omega |x|^{-2a}|\nabla u|^2 dx$. However, according to the method of Brezis and Nirenberg in [1], we can prove that the energy functional satisfies Palais–Smale condition within value range, then use the Mountain Pass Theorem to find a second positive solution by a translated functional. Assume that f satisfies the following conditions:

(f_1) $\lim_{t\to 0^+} \frac{f(x,t)}{t} = +\infty$ and $\lim_{t\to+\infty} \frac{f(x,t)}{t^{p-1}} = 0$ uniformly for $x \in \overline{\Omega}$.

(f_2) $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is nondecreasing with respect to the second variable.

Before stating our main results, we first explain the notations and conventions used throughout this paper. We denote by

$$H := H_0^1(\Omega, |x|^{-2a}).$$

Now we define precisely what we mean by weak solutions to the problem (1).

Definition 1. A function $u \in H$ is said to be a weak solution to (1) if for any $v \in H$, there holds

$$\langle I'(u), v\rangle = \int_\Omega \left(|x|^{-2a}\nabla u \nabla v - \mu \frac{uv}{|x|^{2(1+a)}}\right) dx - \int_\Omega \frac{(u^+)^{p-1}v}{|x|^{bp}} dx - \lambda \int_\Omega f(x,u^+)v dx.$$

Our main results read as follows:

Theorem 1. *Suppose that $N \geq 3, a < \sqrt{\mu}, 0 \leq \mu < (\sqrt{\mu} - a)^2, a \leq b < a + 1, (f_1)$ hold. Then there exists $\lambda^* > 0$ such that the problem (1) has at least one positive weak solution u_λ for any $\lambda \in (0, \lambda^*)$.*

Theorem 2. *Suppose that $N \geq 3, a < \sqrt{\mu}, a \leq b < a + 1, (f_1), (f_2)$ hold. Then there exists $\lambda^* > 0$ such that the problem (1) has at least two positive weak solutions for any $\lambda \in (0, \lambda^*)$ under one of the following conditions:*

(a) $0 \leq \mu \leq (\sqrt{\mu} - a)^2 - \frac{1}{4}, b - a > \frac{N+2}{2N+2}$;

(b) the principle curvatures of $\partial\Omega$ at 0 are negative and $0 \leq \mu \leq (\sqrt{\mu} - a)^2 - (\frac{1}{4} + a^2 - a), a < \frac{N-1}{4}, a < b < a + 1$;

(c) the principle curvatures of $\partial\Omega$ at 0 are negative and $0 \leq \mu \leq (\sqrt{\mu} - a)^2 - (1 + a^2 - 2a), a < \frac{N}{4}, b = a > 0$;

(d) the principle curvatures of $\partial\Omega$ at 0 are non-positive, $0 \leq \mu \leq \min\{(\sqrt{\mu}-a)^2-(\frac{1}{4}+a^2-a), (\sqrt{\mu}-a)^2 - \frac{((1+a-b)p)^2}{4}\}, a < \frac{N-1}{4}, a + \theta < b < a + 1, \theta = \frac{-N^2+9N-6}{6(N-1)}, 4 \leq N \leq 8$;

(e) the principle curvatures of $\partial\Omega$ at 0 are non-positive, $0 \leq \mu \leq \min\{(\sqrt{\mu}-a)^2-(\frac{1}{4}+a^2-a), (\sqrt{\mu}-a)^2 - \frac{((1+a-b)p)^2}{4}\}, a < \frac{N-1}{4}, a < b < a + 1, N \geq 9$;

(f) the principle curvatures of $\partial\Omega$ at 0 are non-positive, $0 \leq \mu \leq \min\{(\sqrt{\mu}-a)^2-(1+a^2-2a), (\sqrt{\mu}-a)^2 - \frac{((1+a-b)p)^2}{4}\}, a < \frac{N}{4}, b = a > 0, N \geq 10$.

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