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Error estimates and superconvergence of a mixed finite element method for elliptic optimal control problems

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ABSTRACT

In this paper, we investigate error estimates and superconvergence of a mixed finite element method for elliptic optimal control problems. The gradient for our method belongs to the square integrable space instead of the classical $H(\mathrm{div};\Omega)$ space. The state and co-state are approximated by the P_0^2–P_1 (velocity–pressure) pair and the control variable is approximated by piecewise constant functions. First, we derive a priori error estimates in H^1-norm for the state and the co-state scalar functions, a priori error estimates in $(L^2)^2$-norm for the state and the co-state vector functions and a priori error estimates in L^2-norm for the control function. Then, using postprocessing projection operator, we derive a superconvergence result for the control variable. Next, we get a priori error estimates in L^2-norm for the state and the co-state scalar functions. Finally, a numerical example is given to demonstrate the theoretical results.

1. Introduction

In the last decades, optimal control problems governed by partial differential equations have been widely studied and applied in the science and engineering numerical simulation. Various numerical methods have been developed to solve these optimal control problems, among them, the finite element approximation of optimal control problems has been extensively studied in the literature. It is impossible to even give a very brief review here. For the studies about convergence and superconvergence of finite element approximations for optimal control problems, see, for example, [1–13]. A systematic introduction of finite element methods for PDEs and optimal control problems can be found in, for example, [14,15].

Although finite element method has successfully simulated a lot of optimal control problems, it fails to solve a certain class of optimal control problems, in which the objective functional contains not only the primal state variable, but also its gradient. For example, in the flow control problem, the gradient stands for Darcy velocity and it is an important physics variable, or, in the temperature control problem, large temperature gradients during cooling or heating may lead to its destruction. Thus, mixed finite element methods will be the best choice for these control problems. Chen et al. have done some works on a priori error estimates and superconvergence properties of Raviart–Thomas mixed finite elements for optimal control problems, see, for example, [16–18]. In [17], Chen used the postprocessing projection operator, which was defined by Meyer and Rösch (see [4]) to prove a quadratic superconvergence of the control by mixed finite element methods. Recently, the authors derived error estimates and superconvergence of mixed methods for convex optimal control problems

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in [18]. In [19], Guo, Fu and Zhang discussed a splitting positive definite mixed finite element method for elliptic optimal control problem and derived a priori error estimates.

In the recent years, Chen et al. [20] developed a new mixed finite element scheme and used P_0^2–P_1 finite element pair for solving partial differential equations. The gradient of the primal variable for this method belongs to the square integrable space instead of the classical $H(\text{div};\Omega)$ space. Using this method, we can derive two approximations for the gradient of the primal variable y, one is the numerical approximation solution $\boldsymbol{p}_h$, the other is the derivative of the approximation solution y_h.

The goal of this paper is to derive a priori error estimates and superconvergence of a new mixed finite element approximation for an elliptic control problem. At first, we will construct new mixed finite element approximation scheme and derive the optimality condition. Then, we introduce some projections and their properties, and prove a priori error estimates for the control variable, the state variables and the co-state variables. Next, we shall derive a superconvergence result for the control variable by using a postprocessing projection operator. At last, we present a numerical experiment to verify the theoretical results. To the best of our knowledge the results are new to the literature concerning a priori error analysis for the new mixed finite element method for elliptic optimal control problems.

We consider the following linear optimal control problems for the state variables $\boldsymbol{p}$, y, and the control u with pointwise control constraint:

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\right\} \tag{1.1}$$

subject to the state equation

$$-\operatorname{div}(A(x)\nabla y)+cy=f+u,\quad x\in\Omega, \tag{1.2}$$

which can be written in the form of the first order system

$$\operatorname{div}\boldsymbol{p}+cy=f+u,\quad \boldsymbol{p}=-A\nabla y,\quad x\in\Omega, \tag{1.3}$$

and the boundary condition

$$y=0,\quad x\in\partial\Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^2$. U_{ad} denotes the admissible set of the control variable, defined by

$$U_{ad}=\{u\in L^2(\Omega):\ a\le u\le b,\ \text{a.e. in }\Omega\}, \tag{1.5}$$

where the bounds $a, b\in\mathbb{R}$ fulfill $a<b$. We assume that $c\ge 0$, $c\in L^\infty(\Omega)$, $y_d\in H^1(\Omega)$, $\boldsymbol{p}_d\in(H^1(\Omega))^2$ and ν is a fixed positive number. The coefficient $A(x)=(a_{ij}(x))$ is a symmetric matrix function with $a_{ij}(x)\in W^{1,\infty}(\Omega)$, which satisfies the ellipticity condition

$$a_*|\xi|^2\le\sum_{i,j=1}^{2}a_{ij}(x)\xi_i\xi_j\le a^*|\xi|^2,\quad\forall(\xi,x)\in\mathbb{R}^2\times\bar{\Omega},\quad 0<a_*<a^*.$$

The plan of this paper is as follows. In Section 2, we construct our new mixed finite element approximation scheme for the optimal control problem (1.1)–(1.4) and give its equivalent optimality conditions. The main results of this paper are stated in Sections 3 and 4. In Section 3, we derive a priori error estimates in H^1-norm for the state and the co-state scalar functions, a priori error estimates in $(L^2)^2$-norm for the state and the co-state vector functions and a priori error estimates in L^2-norm for the control function. In Section 4, using postprocessing projection operator, we derive a superconvergence result for the control variable. Next, we can get a priori error estimates in L^2-norm for the state and the co-state scalar functions. In Section 5, we present a numerical example to demonstrate our theoretical results. In the last section, we briefly summarize the results obtained and some possible future extensions.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p=\sum_{|\alpha|\le m}\|D^\alpha v\|_{L^p(\Omega)}^p$, a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p=\sum_{|\alpha|=m}\|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega)=\{v\in W^{m,p}(\Omega):\ v|_{\partial\Omega}=0\}$. For $p=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, $H_0^m(\Omega)=W_0^{m,2}(\Omega)$, and $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|\cdot\|=\|\cdot\|_{0,2}$. In addition C denotes a general positive constant independent of h, where h is the spatial mesh-size for the control and state discretization.

2. Mixed methods for optimal control problems

In this section, we shall construct our new mixed finite element approximation scheme of the control problem (1.1)–(1.4). For sake of simplicity, we assume that the domain Ω is a convex polygon.

Let

$$\boldsymbol{V}=(L^2(\Omega))^2\ \text{ and }\ W=H_0^1(\Omega). \tag{2.1}$$

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