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# Grammian-type determinant solutions to generalized KP and BKP equations

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ABSTRACT

Grammian-type determinant identities of the bilinear KP hierarchy and a useful property on the derivatives of Grammians are proposed. Based on these Grammian-type determinant identities, three Grammian structures are furnished to a large class of generalized nonlinear KP and BKP type equations. All the generating functions for matrix entries satisfy a system of combined linear partial differential equations. In the first case of the class of generalized KP and BKP equations, general rational solutions are given, making use of its Grammian formulation.



## 1. Introduction

It is significantly important in the studies of the nonlinear phenomena to search for multi-soliton solutions to nonlinear evolution equations (NLEEs). The bilinear method first proposed by Hirota in 1971 [1] plays a vital role in constructing soliton solutions related to bilinear equations. There exist several different types with respect to the form of the $N$-soliton solutions. The $N$-soliton solutions may be expressed either by the polynomials of various exponentials [2], or by the Wronskian determinants [3–5], or by the Grammian determinants [6,7]. It was revealed that solitons and rational solutions could be written as Wronskian determinants to many integrable equations, such as the Korteweg–de Vries (KdV) equation [8,9] and the Boussinesq equation [4,10]. The Grammian determinant is as the determinant of a Gram matrix whose matrix entries are in the integral expression. Typical higher-dimensional soliton equations, including the $(3+1)$-dimensional generalized Kadomtsev–Petviashvili (KP) equation [7], the $(3+1)$-dimensional Jimbo–Miwa equation [11–14] and the Davey–Stewartson (DS) equation [15,16] possess Grammian determinant solutions based on their Hirota bilinear forms. The Jacobi identity for determinants is the key to construct the Grammian formulation. Recently, Nimmo and Zhao described the Grammian-type determinant identity of the KP equation via a standard partition notation in Ref. [17].

In this paper, we focus on a large class of generalized KP and BKP type equations

$$\begin{aligned} &u_{xxxy} + 3(u_x u_y)_x + c_1 u_{xy} + c_2 u_{xz} + c_3 u_{xt} + c_4 u_{yz} + c_5 u_{yt} \\ &\quad + c_6 u_{xx} + c_7 u_{yy} + c_8 u_{zz} = 0,\ \ c_i \in \mathbb{R}, i = 1, 2, \ldots, 8, \end{aligned} \tag{1}$$

which can be widely applied in fluid mechanics, plasma physics, chemistry, biology and other fields. Let us recall Hirota bilinear derivatives introduced by the following rule [18]:

$$D_{x_1}^{n_1} D_{x_2}^{n_2} f \cdot g = (\partial_{x_1} - \partial_{x_1'})^{n_1} (\partial_{x_2} - \partial_{x_2'})^{n_2} f(x_1, x_2) g(x_1', x_2')|_{x_1'=x_1, x_2'=x_2},$$

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where $n_1$ and $n_2$ are arbitrary nonnegative integers. Through the dependent variable transformation

$$u = 2(\ln f)_x, \tag{2}$$

every nonlinear equation defined by (1) can be written as

$$\begin{aligned} P(D_x, D_y, D_z, D_t)f \cdot f &= (D_x^3 D_y + c_1 D_x D_y + c_2 D_x D_z + c_3 D_x D_t \\ &\quad + c_4 D_y D_z + c_5 D_y D_t + c_6 D_x^2 + c_7 D_y^2 + c_8 D_z^2)f \cdot f = 0, \end{aligned} \tag{3}$$

which is equivalent to

$$\begin{aligned} &(f_{xxxy} + c_1 f_{xy} + c_2 f_{xz} + c_3 f_{xt} + c_4 f_{yz} + c_5 f_{yt} + c_6 f_{xx} + c_7 f_{yy} \\ &\quad + c_8 f_{zz})f - f_{xxx} f_y + 3 f_{xx} f_{xy} - 3 f_x f_{xxy} - c_1 f_x f_y - c_2 f_x f_z \\ &\quad - c_3 f_x f_t - c_4 f_y f_z - c_5 f_y f_t - c_6 f_x^2 - c_7 f_y^2 - c_8 f_z^2 = 0. \end{aligned} \tag{4}$$

As applications of Grammian-type determinant identities, the purpose of this paper is to establish Grammian determinant solutions for some special cases of the class of generalized nonlinear equations (1). A group of sufficient conditions consisting of systems of combined linear partial differential equations are given which guarantee Grammian determinants solve the generalized bilinear KP and BKP equations.

## 2. Grammian-type determinant identities

The fundamental classification of integrable equations in Hirota bilinear form contains the hierarchy of the KP equation [19]

$$(D_1^4 - 4D_1 D_3 + 3D_2^2)f \cdot f = 0, \tag{5a}$$

$$[(D_1^3 + 2D_3)D_2 - 3D_1 D_4]f \cdot f = 0, \tag{5b}$$

$$\ldots.$$

Here $f$ is a function with regard to variables $x_k$, $k = 1, 2, 3, \ldots$ and $D_k \equiv D_{x_k}$. Now we consider the following Grammian determinant:

$$G = \det(a_{ij})_{1\le i,j\le N}, \ a_{ij} = c_{ij} + \int^x \phi_i \psi_j dx, \ c_{ij} = \text{constant}, \tag{6}$$

where functions $\phi_i$ and $\psi_j$ depending on variables $x_1(= x), x_2, x_3, \ldots$, satisfy

$$\frac{\partial \phi_i}{\partial x_k} = \frac{\partial^k \phi_i}{\partial x^k} = \phi_i^{(k)}, \ k = 1, 2, 3, \ldots, \ 1 \le i \le N, \tag{7}$$

$$\frac{\partial \psi_j}{\partial x_k} = (-1)^{k+1} \frac{\partial^k \psi_j}{\partial x^k} = (-1)^{k+1} \psi_j^{(k)}, \ k = 1, 2, 3, \ldots, \ 1 \le j \le N, \tag{8}$$

and $\phi_i = \phi_i^{(0)}$, $\psi_j = \psi_j^{(0)}$.

Firstly, let us characterize the useful result on the bilinear KP hierarchy as follows:

**Lemma 2.1** ([*14,18*]). *Let $\phi_i$ and $\psi_j$ satisfy (7) and (8), respectively. Then the Grammian determinant $f = G = \det(a_{ij})_{1\le i,j\le N}$ defined by (6) solves the bilinear KP hierarchy equations (5a) and (5b).*

Lemma 2.1 has been proved by Hirota and Wu [14,18] in the form of Grammian determinants which can be expressed by a Pfaffian structure.

To obtain Grammian-type determinant identities of the bilinear KP hierarchy, we introduce the shorthand notation defined by Nimmo and Zhao [17], who set

$$\partial_\lambda G \equiv \frac{\partial^p G}{\partial x_{\lambda_1} \ldots \partial x_{\lambda_p}}, \tag{9}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ is a sequence of positive integers $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_p > 0$. Using the result for differentiating KP Grammians, a direct calculation tells that

$$\begin{aligned} (D_1^4 - 4D_1 D_3 + 3D_2^2)G \cdot G &= (-4\partial_{(31)}G + 3\partial_{(2^2)}G + \partial_{(1^4)}G)G \\ &\quad - 4(-\partial_{(3)}G + \partial_{(1^3)}G)\partial_{(1)}G + 3(\partial_{(1^2)}G)^2 - 3(\partial_{(2)}G)^2 \end{aligned}$$