Computers and Mathematics with Applications 74 (2017) 744–758

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Backwards compact attractors and periodic attractors for non-autonomous damped wave equations on an unbounded domain

Jinyan Yin [a,b,*], Yangrong Li [a], Anhui Gu [a]

[a] School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
[b] School of Mathematics and Information, China West Normal University, Nanchong 637002, China

ARTICLE INFO

Article history:
Received 27 February 2017
Received in revised form 11 May 2017
Accepted 13 May 2017
Available online 2 June 2017

Keywords:
Backwards compact attractors
Periodic attractors
Pullback attractors
Non-autonomous wave equations
Unbounded domains

ABSTRACT

We study the backwards dynamics for the wave equation defined on the whole 3D Euclid space with a positively bounded coefficient of the damping and a time-dependent force. We introduce a backwards compact attractor which is the minimal one among the backwards compact and pullback attracting sets. We prove that a backwards compact attractor is equivalent to a pullback attractor (invariant) that is backwards compact, i.e. the union of the attractor over the past time is pre-compact. We also establish a sufficient and necessary criterion of the existence of a backwards compact attractor and show the relationship of a periodic attractor. As an application of these abstract results, we prove that the non-autonomous wave equation has a backwards compact attractor under some backwards assumptions of the non-autonomous force. Moreover, we establish the backwards compactness from some periodicity assumptions, more precisely, if the force is assumed only to be periodic then a backwards compact attractor exists, and if the damped coefficient is further assumed to be periodic then the attractor is both periodic and backwards compact.

1. Introduction

In this paper, we consider the backwards compact dynamics for the following non-autonomous damped wave equation on $\mathbb{R}^3$:

$$\begin{cases} u_{tt}+\beta(t)u_t-\Delta u+\lambda u+f(x,u)=g(x,t), & x\in\mathbb{R}^3,\ t\geq\tau, \\ u(x,\tau)=u_0, \qquad u_t(x,\tau)=u_1, & x\in\mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\tau\in\mathbb{R}$, $\lambda>0$, the varying coefficient β is positive and bounded, the nonlinearity $f\in C^1(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$ satisfies the same growth conditions as given in [1,2], and the time-dependent force g will be special.

The nonlinear wave equation arises in relativistic quantum mechanics, which has great physical significance and receives many attentions. For instance, the asymptotic dynamics were established in [1–14] for deterministic or stochastic wave equations.

In particular, the authors in [1,2,15] studied the existence of random attractors for the stochastic equation with a constant coefficient $\beta(\cdot)\equiv c>0$. When the coefficient is time-dependent but the equation is restricted in a bounded domain, the authors in [5,7] proved that the evolution process had a pullback attractor $\mathcal{A}$ that is *backwards bounded*, i.e. $\bigcup_{s\leq t}\mathcal{A}(s)$ is bounded for each $t\in\mathbb{R}$.

* Corresponding author at: School of Mathematics and Statistics, Southwest University, Chongqing 400715, China.
E-mail addresses: yjy111@email.swu.edu.cn (J. Yin), liyr@swu.edu.cn (Y. Li), gahui@swu.edu.cn (A. Gu).

http://dx.doi.org/10.1016/j.camwa.2017.05.015
0898-1221/

Our main purposes in this paper are to prove that the evolution process derived by Eq. (1.1) possesses a pullback attractor that is *backwards compact*, i.e. both $\mathcal{A}(t)$ and $\overline{\bigcup_{s\le t}\mathcal{A}(s)}$ are compact for each $t \in \mathbb{R}$, and to establish the relationship between such an attractor and a periodic attractor. Note that, the compactness investigated here is distinguished from the usual section compactness in many literatures, i.e. every time-section is compact (see [2,7,8,11,14,16–18]).

To achieve our goals, we introduce the concept of a *backwards compact attractor* that is the minimal one among the backwards compact and pullback attracting sets. We show that a backwards compact attractor is equivalent to a pullback attractor that is backwards compact, and thus it must be an invariant set. One can also refer to [19,20] to see some discussions about the backwards compactness of a pullback attractor.

We then establish a sufficient and necessary criterion of the existence of a backwards compact attractor for a non-compact system. We prove that a *backwards pullback asymptotically compact* evolution process has a backwards compact attractor if and only if it has an *increasing*, bounded and pullback absorbing set, or it has a bounded and *backwards pullback absorbing set* (see Theorem 2.8). We also show that a periodic attractor must be a backwards compact attractor (see Theorem 2.10).

To ensure the above abstract criteria can be applied to the non-autonomous wave equation, we need to make some new (but natural) hypotheses on the force g. A basic assumption is the so-called *backwards translation boundedness*, which appeared in [19]. Under this basic assumption, we can prove the derived evolution process has an increasing, bounded and pullback absorbing set. If we further assume that g is *backwards tail-small* (see (3.9)) and *backwards complement-small* (see (3.10)), then we can prove the backwards pullback asymptotic compactness by using a spectrum decomposition method together with a backwards uniform tail-estimate. Therefore, the theoretical result can be applied to show that the evolution process induced by Eq. (1.1) possesses a backwards compact attractor in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ (see Theorem 4.4).

In the final section, we consider the relationship between the periodicity and backwards compactness of a pullback attractor for Eq. (1.1). The authors in [2,21] proved that the attractor $\mathcal{A}$ is periodic if the force g is periodic (note that the coefficient β in [2,21] is a constant and thus it is periodic). In this paper, we prove that the periodicity of g implies the three assumptions mentioned above. Therefore, we obtain a backwards compact attractor in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ if we assume only g is periodic (it does not require the periodicity assumption of the coefficient β). Furthermore, the attractor is both periodic and backwards compact if β is also assumed to be periodic.

2. Abstract results on backwards compact attractors

In this section, we review some relevant notions of a pullback attractor (see [5,7,22–24]), introduce the concept of a backwards compact attractor and then investigate its existence and the relationship of a periodic attractor.

2.1. The existence result

Let $(X, \|\cdot\|_X)$ be a Banach space. A set-valued mapping $\mathcal{D} : \mathbb{R} \to 2^X \setminus \{\emptyset\}$ is called a non-autonomous set in X. A non-autonomous set $\mathcal{D}$ is called increasing if $\mathcal{D}(s) \subset \mathcal{D}(t)$ for all $s \le t$, and $\mathcal{D}$ is said to have a topological property (such as closedness, boundedness or compactness) if $\mathcal{D}(t)$ has this property with each $t \in \mathbb{R}$.

Definition 2.1. A non-autonomous set $\mathcal{D} \subset X$ is called backwards compact if both $\mathcal{D}(t)$ and $\overline{\bigcup_{s\le t}\mathcal{D}(s)}$ are compact in X for each $t \in \mathbb{R}$, and it is called backwards bounded if $\bigcup_{s\le t}\mathcal{D}(s)$ is bounded in X for each $t \in \mathbb{R}$.

Definition 2.2. An evolution process S in X is a family of mappings $S(t, r) : X \to X$ with $t \ge r$, which enjoys

$$S(r, r) = id_X, \qquad S(t, r) = S(t, s)S(s, r), \quad \text{for all } t \ge s \ge r \text{ with } t \in \mathbb{R}, \text{ and} \tag{2.1}$$

$$(t, r, x) \to S(t, r)x \text{ is continuous with } t \ge r \text{ and } x \in X. \tag{2.2}$$

Definition 2.3 (*[7]*). A non-autonomous set $\mathcal{A} \subset X$ is called a pullback attractor for an evolution process S if

(i) $\mathcal{A}(t)$ is compact for each $t \in \mathbb{R}$;

(ii) $\mathcal{A}$ is invariant, i.e. $S(t, r)\mathcal{A}(r) = \mathcal{A}(t)$ for all $t \ge r$;

(iii) $\mathcal{A}$ pullback attracts each bounded subset $B \subset X$, i.e. for each $t \in \mathbb{R}$,

$$\lim_{\tau\to\infty} \text{dist}(S(t, t-\tau)B, \mathcal{A}(t)) = 0,$$

where $\text{dist}(A, B) := \sup_{a\in A}\inf_{b\in B}\|a - b\|_X$ $(A, B \subset X)$ is the Hausdorff semi-distance in X;

(iv) $\mathcal{A}$ is the minimal closed pullback attracting set.

Definition 2.4. A non-autonomous set $\mathcal{A} \subset X$ is called a *backwards compact attractor* for an evolution process S if

(i) $\mathcal{A}$ is backwards compact;

(ii) $\mathcal{A}$ pullback attracts each bounded subset $B \subset X$;

(iii) $\mathcal{A}$ is the minimal one with properties (i) and (ii).

In fact, the minimality in Definition 2.4 indicates the invariance. Therefore, a backwards compact attractor must be a pullback attractor.