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A short FE implementation for a 2d homogeneous Dirichlet problem of a fractional Laplacian

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ABSTRACT

In Acosta et al. (2017), a complete n-dimensional finite element analysis of the homogeneous Dirichlet problem associated to a fractional Laplacian was presented. Here we provide a comprehensive and simple 2D *MATLAB*® finite element code for such a problem. The code is accompanied with a basic discussion of the theory relevant in the context. The main program is written in about 80 lines and can be easily modified to deal with other kernels as well as with time dependent problems. The present work fills a gap by providing an input for a large number of mathematicians and scientists interested in numerical approximations of solutions of a large variety of problems involving nonlocal phenomena in two-dimensional space.

1. Introduction

The Finite Element Method (FEM) is one of the preferred numerical tools in scientific and engineering communities. It counts with a solid and long established theoretical foundation, mainly in the linear case of second order *elliptic* partial differential equations. These kinds of operators, with the Laplacian as a canonical example, are involved in modeling *local* diffusive processes. On the other hand, nonlocal or *anomalous* diffusion models have increasingly impacted upon a number of important areas in science. Indeed, non-local formulations can be found in physical and social contexts, modeling as diverse phenomena as human locomotion in relation to crime diffusion [1], electrodiffusion of ions within nerve cells [2] or machine learning [3].

The Fractional Laplacian (FL) is among the most prominent examples of a non-local operator. For $0 < s < 1$, it is defined as

$$(-\Delta)^s u(x) = C(n,s) \quad \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy, \tag{1.1}$$

where

$$C(n,s) = \frac{2^{2s} s \Gamma\left(s+\frac{n}{2}\right)}{\pi^{n/2}\Gamma(1-s)}$$

is a normalization constant. The FL, given by (1.1), is one of the simplest pseudo-differential operators and can also be regarded as the infinitesimal generator of a $2s$-stable Lévy process [4].

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Given a function f defined in a bounded domain Ω, the homogeneous Dirichlet problem associated to the FL reads: find u such that

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{1.2}$$

In contrast to elliptic PDEs, numerical developments for problems involving this non-local operator, even in simplified contexts, are seldom found in the literature. The reason for that is related to two major challenging tasks usually involved in its numerical treatment: the handling of highly singular kernels and the need to cope with an unbounded region of integration. This is precisely the case of (1.2), for which just a few numerical methods have been proposed. Effectively implemented *in one space dimension*, we mention, for instance: a finite difference scheme by Huang and Oberman [5], a FE approach developed by D'Elia and Gunzburger [6] that relies on a volume-constrained version of the non-local operator and a simple one-dimensional spectral approach [7]. We refer the reader to [8] for a more detailed account of these schemes and a discussion on other fractional diffusion operators on bounded domains and their discretizations.

To the best of the authors' knowledge, numerical computations for (1.2) in higher dimensions have become available only recently [8]. In that paper a complete n-dimensional finite element analysis for the FL has been carried out, including regularity of solutions of (1.2) in standard and weighted fractional spaces. Moreover, the convergence for piecewise linear elements is proved with optimal order for both uniform and graded meshes.

In that work there are presented error bounds in the energy norm and numerical experiments (in 2D), demonstrating an accuracy of the order of $h^{1/2}\log h$ and $h\log h$ for solutions obtained by means of uniform and graded meshes, respectively.

The present article can be seen as a complementary work to [8], providing a short and simple *MATLAB*® FE code coping with the homogeneous Dirichlet problem (1.2).

In [9] a *MATLAB*® implementation for *linear* finite elements and *local* elliptic operators is presented in a concise way. We tried to emulate as much as possible that spirit in the non-local context. Notwithstanding that and in spite of our efforts, some intrinsic technicalities make our code inevitably slightly longer and more complex than that. Just to give a hint about it, we take a glimpse in advance at the nonlocal stiffness matrix K. It involves expressions of the type

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{(\varphi_i(x)-\varphi_i(y))(\varphi_j(x)-\varphi_j(y))}{|x-y|^{2+2s}}\,dxdy, \tag{1.3}$$

where φ_i, φ_j are arbitrary nodal basis functions associated to a triangulation $\mathcal{T}$. Two difficulties become apparent in the calculation of (1.3). First, at the element level, computing (1.3) leads to terms like

$$\int_T\int_{\tilde{T}} \frac{(\varphi_i(x)-\varphi_i(y))(\varphi_j(x)-\varphi_j(y))}{|x-y|^{2+2s}}\,dxdy, \tag{1.4}$$

for arbitrary pairs $T, \tilde{T} \in \mathcal{T}$. If T and $\tilde{T}$ are not neighboring then the integrand in (1.4) is a regular function and can be integrated numerically in a standard fashion. On the other hand, if $T \cap \tilde{T} \neq \emptyset$ an accurate algorithm to compute (1.4) is not easy to devise. Fortunately, (1.4) bears some resemblances to typical integrals appearing in the Boundary Element Method [10] and we extensively exploit this fact. Indeed, a basic and well known technique in the BEM community is to rely on Duffy-type transforms. This approach leads us to the decomposition of such integrals into two parts: a highly singular but explicitly integrable part and a smooth, numerically treatable part. We use this method to show how (1.4) can be handled with an arbitrary degree of precision (this is carefully treated in Appendices A.1, A.2, A.3, A.4).

Yet another difficulty is hidden in the calculation of K. Although Ω is a bounded domain and the number of potential unknowns is always finite, (1.3) involves a computation in $\mathbb{R}^2 \times \mathbb{R}^2$. In particular, in the homogeneous setting, we need to accurately compute the function

$$\int_{\Omega^c} \frac{1}{|x-y|^{2+2s}}\,dy, \tag{1.5}$$

for any $x \in \Omega$. That, of course, can be hard to achieve for a domain with a complex boundary. Nonetheless, introducing an extended secondary mesh, as it is explained in Section 3, it is possible to reduce such problem to a simple case in which $\partial\Omega$ is a circle. We show that in this circumstance a computation of (1.5) can be both fast and accurately delivered (see also Appendix A.5). Remarkably, this simple idea applies in arbitrary space dimensions.

Regarding the code itself, our main concern has been to keep a compromise between readability and efficiency. First versions of our code were plainly readable but too slow to be satisfactory. In the code offered here many computations have been vectorized and a substantial speed up gained, sometimes at the price of losing (hopefully not too much) readability.

Last but not least, the full program is available from the authors upon request, so that the reader can avoid retyping it. Small modifications of the base code may make it usable for dealing with many different problems. It has been successfully used in several contexts such as eigenvalue computations and time dependent problems (considering semi and full fractional settings), among others.

The paper is organized as follows. In Section 2, we review appropriate fractional spaces and regularity results for (1.2). Section 3 deals with basic aspects of the FE setting. The data structure is carefully discussed in Section 4 and the main loop