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A new fourth-order compact scheme for the Navier–Stokes equations in irregular domains

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ABSTRACT

We present a high-order finite difference scheme for Navier–Stokes equations in irregular domains. The scheme is an extension of a fourth-order scheme for Navier–Stokes equations in streamfunction formulation on a rectangular domain (Ben-Artzi et al., 2010). The discretization offered here contains two types of interior points. The first is regular interior points, where all eight neighboring points of a grid point are inside the domain and not too close to the boundary. The second is interior points where at least one of the closest eight neighbors is outside the computational domain or too close to the boundary. In the second case we design discrete operators which approximate spatial derivatives of the streamfunction on irregular meshes, using discretizations of pure derivatives in the x, y and along the diagonals of the element.

1. Introduction

In this paper we are interested in high-order discretizations of the Navier–Stokes equations. The Navier–Stokes equations play a central role in modeling fluid flows. Here we focus on incompressible flows. It is well-known that this system may be represented in pure streamfunction formulation as follows (see [1,2]).

$$\begin{cases} \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \nu\Delta^2\psi = f(x,y,t), \\ \psi(x,y,t) = \psi_0(x,y). \end{cases} \tag{1.1}$$

Recall that $\nabla^{\perp}\psi = (-\partial_y\psi, \partial_x\psi)$ is the velocity vector. The no-slip boundary condition associated with this formulation is

$$\psi = \frac{\partial\psi}{\partial n} = 0, \quad (x,y) \in \partial\Omega,\ t > 0 \tag{1.2}$$

and the initial condition is

$$\psi(x,y,0) = \psi_0(x,y), \quad (x,y) \in \Omega. \tag{1.3}$$

In this paper we extend the fourth-order scheme [3] to irregular domains. The strategy used here is to present the biharmonic operator $\partial_x^4 + 2\partial_x^2\partial_y^2 + \partial_y^4$ as a combination of pure fourth-order derivatives in the x, y and the diagonal directions $\eta = (x+y)/\sqrt{2},\ \ \xi = (y-x)\sqrt{2}$. Then, the pure fourth-order derivatives may be approximated via a compact scheme

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using the values of the function and its directional derivatives (see also [4,5]). An alternative approach is to construct a two-dimensional polynomial which collocates the values of the function and its directional derivatives at the corners of the irregular element and then approximate the biharmonic of the function by the biharmonic of this polynomial (see [6]).

The numerical resolution of the Navier–Stokes system, governing viscous, incompressible, time-dependent flow, has been an important challenge of computational fluid dynamics. References belonging to the class of finite difference methods for the approximation of Navier–Stokes equations include projection methods [7–11]. The pure-streamfunction formulation for the time-dependent Navier–Stokes system in planar domains has been used in [12–14] some twenty years ago. It has been designed primarily for the numerical investigation of the Hopf bifurcation occurring in the driven cavity problem. Their approach was based on a finite-difference method. The application of various compact schemes to the pure streamfunction formulation is fairly recent [15–19].

We review some numerical methods for irregular domains. There are many references for finite elements methods for irregular domain (see for the example [20,21]). Several references for finite difference methods include [22–24]. In [24] a six-point scheme (star) was suggested. The disadvantage of the latter is its singularity and ill-conditioning. Several references such as [25] use coordinates transformation, however this approach is not suited to multiple irregular boundaries and may also impose singularities due to the coordinate transformation.

Liszka and Orkisz stated in [26] (1980) that "The fascination for FEM, however, caused by enormous successes or simply by fashion, has resulted in a relative stagnation in some other methods, especially in finite difference methods". In [26] a new mesh generation was constructed.

In [27,5] parabolic equations (in particular the heat equation) were solved in irregular domain, where a cartesian grid was used to approximate the solution of a time-dependent diffusion problem. At near boundary points the derivatives ∂_x^2 and ∂_y^2 were approximated using a non-uniform mesh, where one of the neighbors of the computational point was taken as a boundary point. In [28] Colella et al. suggested an embedded boundary/volume method for Navier–Stokes equations in irregular domains. It is a combination of finite differences, embedded boundary algorithm and finite volume methods. Calhoun [29] approximates the vorticity–streamfunction formulation by adding a correction term to the Poisson equation (which relates the streamfunction to the vorticity) using the immersed interface method. The purpose of this correction is to impose both boundary conditions on the streamfunction and the singular sources for the vorticity equation. The numerical results show second-order convergence rates for the solution of the Navier–Stokes equations. In [30] a fast finite difference method is proposed to solve the incompressible Navier–Stokes equations on a general domain. The method is based on the vorticity stream-function formulation and a fast Poisson solver defined on a general domain using the immersed interface method.

In [31] the discretization of the Poisson equation on irregular domains at near boundary points was carried out via quadratic polynomials, which yields second-order accuracy of the scheme. In [32] second and fourth order Cartesian grid finite difference methods were developed for second order elliptic and parabolic partial differential equations on irregular domains. The information around an irregular point was completed via a two-dimensional Taylor expansion around a boundary point using a local coordinate system. In [33] the immersed interface method is invoked for the application of the boundary conditions to the velocity–pressure formulation of Navier–Stokes equations. The approximated rates of convergence are between 2 and 3. In [34] the Poisson equation which relates the streamfunction to the vorticity was solved in two steps in order to enhance the efficiency of the scheme.

In [6,2] a two-dimensional interpolating polynomial of degree 5 and a half was constructed to approximate the solution of the biharmonic problem. This polynomial collocates the values of the function and its directional derivatives at the corners of an irregular element near the boundary (as well as of regular inner elements) and then approximates the biharmonic of the solution by the biharmonic of this polynomial. Fourth-order accuracy was achieved for the biharmonic problem in a circle and an ellipse.

In this paper we approximate spatial derivatives of the Navier–Stokes equations in streamfunction formulation. Interior points are treated via fourth-order discretizations as in [3]. Irregular elements are formed near the boundary, as in [6]. For irregular elements we write the biharmonic operator, as well as the convective term, using pure derivatives only in the directions of the axis and the diagonals of the element. Then, one-dimensional interpolating operators are used for these elements. Note that in [35] we have proved that the solution of the one-dimensional biharmonic equation by our compact high-order scheme is fourth-order accurate. Thus, it may be proved that reduction to our scheme to one dimension is fourth-order accurate (see also [36]).

The outline of the paper is as follows. In Section 2 we describe fourth order approximations of the Navier–Stokes equations in regular domains. All spatial operators appearing in the evolution equation, i.e., the Laplacian, the biharmonic operator and the nonlinear convective term, are approximated via fourth-order schemes. We also describe a time-marching scheme for the temporal evolution.

In Section 3 we suggest a new scheme for the Navier–Stokes system in streamfunction formulation for irregular domains. Here we assign different flags to the cartesian grid points in the rectangle, in which the irregular domain is embedded. At near-boundary points we approximate the spatial operators via combinations of pure spatial derivatives in the directions of the axis x, y and the diagonals.

In Section 4 we detail the approximations of $\partial_x \psi$, $\partial_x^4 \psi$ and $\partial_x^2 \psi$ for an irregular element. Similar representations are valid to $\partial_y \psi$, $\partial_y^4 \psi$ and $\partial_y^2 \psi$. The fourth-order derivatives along the diagonals, $\partial_\eta^4 \psi$ and $\partial_\xi^4 \psi$, are approximated in the same fashion. In Section 5 we describe the approximation of the convective term at near boundary points. This involves discretizations

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