Computers and Mathematics with Applications 73 (2017) 2345–2359

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A dispersion minimizing finite difference scheme for the Helmholtz equation based on point-weighting[☆]

Dongsheng Cheng[a], Xu Tan[a], Taishan Zeng[b,*]

[a] School of Software Engineering, Shenzhen Institute of Information Technology, Shenzhen 518172, PR China
[b] School of Mathematical Sciences, South China Normal University, Guangzhou 510631, PR China

ARTICLE INFO

Article history:
Received 20 May 2016
Received in revised form 4 April 2017
Accepted 9 April 2017
Available online 29 April 2017

Keywords:
Helmholtz equation
Perfectly matched layer
Finite difference scheme
Numerical dispersion

ABSTRACT

In this paper, we develop a new dispersion minimizing finite difference scheme for the Helmholtz equation with perfectly matched layer (PML) in two dimensional domain, which is a second order 9-point scheme. To discretize the second derivative operator, we employ a linear combination of a point and its neighboring grid points to replace each of the five points in the traditional central difference scheme. Based on minimizing the numerical dispersion, the combination weights are determined by minimizing the numerical dispersion with a flexible selection strategy. The new scheme is simple, rotation-free, and pointwise consistent with the equation, which is different from the classical rotated 9-point difference scheme obtained by combining the Cartesian coordinate system and the rotated system. Moreover, it is a robust scheme even if the step sizes of different directions are not equal. Convergence analysis and dispersion analysis are given. Several numerical examples are presented to illustrate the numerical convergence and effectiveness of the new scheme.

1. Introduction

Helmholtz equation is very important in many fields of science and engineering, for instance, in aeronautics, marine technology, geophysics and optical problems. The Helmholtz equation is so important that its numerical simulation has attracted significant research interest. To simulate the Helmholtz equation numerically, artificial boundary conditions are often employed to truncate the infinite computing domain into a finite one. Two popular artificial boundary conditions are the perfectly matched layer (PML, cf. [1–3]) and absorbing boundary condition (cf. [4,5]). In practice, the PML has excellent absorbing performance. It generates almost no reflection at the interface between the interior medium and the artificial absorbing medium. In this paper, we consider the problem of solving the Helmholtz equation with PML in the two dimensional domain.

To solve the Helmholtz equation, we mainly have finite element methods (cf. [6–12]) and finite difference methods (cf. [13–19]). It is known that all the numerical methods suffer from the "pollution effect" (cf. [8,20]), which can never be eliminated for the two dimensional (2D) and three dimensional (3D) cases. Because of the pollution effect, the wavenumber

☆ Supported in part by the Natural Science Foundation of Guangdong Province under grants 2015A030313373 and 2015A030313589, by the Science and Technology Innovation Commission of Shenzhen under grants JCYJ20150630114942315 and JCYJ20160527102119211, by the Characteristic Innovation Project from the Educational Department of Guangdong Province under grant 2016KTSCX024, by the Natural Science Foundation of China under grant 11371152.

* Corresponding author.
E-mail addresses: chengds@sziit.edu.cn (D. Cheng), tanx@sziit.edu.cn (X. Tan), zengtsh@m.scnu.edu.cn (T. Zeng).

http://dx.doi.org/10.1016/j.camwa.2017.04.005

of the numerical solution is different from the wavenumber of the exact solution, which is known as the "numerical dispersion" (cf. [20,21]). Due to the numerical dispersion, the numerical method requires a finer mesh to ensure the accuracy with the increasing wavenumber. To suppress the pollution effect and reduce the numerical dispersion, a number of numerical methods (cf. [6,7,22,14–16,23,19,24]) are proposed during the past few decades.

Finite difference methods have been widely used in solving the Helmholtz equation, especially in the engineering field such as oil–gas exploration. It is known that the standard 2D 5-point finite difference scheme leads to serious numerical dispersion. To reduce the numerical dispersion, a rotated 9-point finite difference scheme (cf. [14]) was developed by linearly combining the two discretizations of the equation on the classical Cartesian coordinate system and the 45° rotated system. In [16], the rotated 9-point difference scheme was extended to a 25-point scheme, which gave a better performance. However, the bandwidth of the resulting matrix is much wider than that of the 9-point scheme. In [25–27], Chen et al. developed the derivative-weighting difference schemes, which are rotation-free and consistent with the Helmholtz equation. However, the derivative-weighting difference scheme does not work when the step sizes of different directions are not equal, which is usually the case in practical applications. In [23], a dispersion minimizing difference scheme was constructed for the 3D Helmholtz equation based on the ray theory. In [19], an exact finite difference scheme was proposed, which required to solve the equation repeatedly in order to obtain a good solution. To reduce the numerical error, higher-order finite difference schemes (cf. [13,17,18,28]) were also commonly used. However, the higher-order schemes require the source term (right-hand side) to be smooth enough, which cannot be satisfied in many cases.

In this paper, a new dispersion minimizing finite difference scheme is proposed for the 2D Helmholtz equation, which is a second order scheme. To construct the scheme, we first employ the standard 5-point central difference scheme to discretize the second derivative operator and the zeroth term of the Helmholtz equation, and then replace each of the five grid points by a weighted combination of the point and its three neighboring points. This scheme is different from the derivative-weighting one, we call it the point-weighting scheme. The weights (parameters) are obtained by minimizing the numerical dispersion via the dispersion relation formula, and a flexible selection strategy is suggested in the determination of the weights, which makes full use of the priori information. The convergence analysis and dispersion analysis are given. Compared with the rotated difference scheme, the new scheme is simpler, more effective, and is pointwise consistent with the Helmholtz equation. In comparison with the derivative-weighting scheme, the new scheme has much better performance in reducing the numerical dispersion in the case that the step sizes are not equal in different directions. Hence, the new scheme is a robust one, and we demonstrate this both theoretically and numerically.

This paper is organized as follows. In Section 2, we propose a new dispersion minimizing finite difference scheme for the 2D Helmholtz equation with PML based on point weighting, and prove that the new scheme is second order in accuracy. In Section 3, numerical dispersion analysis is given, and a selection strategy is introduced to determine the weights in the new scheme. In Section 4, numerical experiments are given to validate numerical convergence and effectiveness of the new scheme. Finally, in Section 5, some conclusions are drawn.

2. A new finite difference scheme based on point weighting

In this section we develop a new finite difference scheme for the 2D Helmholtz equation with PML based on point weighting, and present the convergence analysis. Moreover, we point out that the new scheme is pointwise consistent with the Helmholtz equation with PML.

We start with describing the Helmholtz equation with PML [1,3]. Consider the Helmholtz equation

$$\mathscr{A}u := -\Delta u - k^2 u = g \quad \text{in } \mathbb{R}^2, \tag{2.1}$$

where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian, k is the wavenumber defined as $k := 2\pi f/v$ with f indicating the frequency and v indicating the speed, u is the unknown representing a pressure field, and the right side g represents the source term. The wavenumber k is a constant for the homogeneous medium, and varies for the heterogeneous medium.

Applying PML technique to truncate the infinite domain of Eq. (2.1) into a bounded domain leads to the equation

$$-\frac{\partial}{\partial x}\left(\frac{e_y}{e_x}\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{e_x}{e_y}\frac{\partial u}{\partial y}\right) - e_x e_y k^2 u = g, \tag{2.2}$$

where $e_x := 1 - i\frac{\sigma_x}{\omega}$ and $e_y := 1 - i\frac{\sigma_y}{\omega}$, in which $\omega := 2\pi f$ is the angular frequency, σ_x and σ_y are usually chosen as differentiable functions only depending on the variables x and y respectively for the end of reducing the numerical reflection. Generally, σ_x is defined as

$$\sigma_x := \begin{cases} 2\pi a_0 f_0 \left(\dfrac{l_x}{L_{PML}}\right)^2, & \text{inside } PML, \\ 0, & \text{outside } PML, \end{cases} \tag{2.3}$$

where f_0 is the dominant frequency of the source, L_{PML} is the thickness of PML, l_x is the distance from interface between the interior region and PML region. Moreover, a_0 is a constant, and we choose $a_0 = 1.79$ according to [3]. σ_y can be chosen similarly. In interior region, $e_x \equiv 1$, $e_y \equiv 1$. The wavelength is defined by $\lambda := v/f$, and the number of wavelengths in a