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A blow-up result for a nonlinear damped wave equation in exterior domain: The critical case

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ABSTRACT

We consider the initial boundary value problem of the nonlinear damped wave equation in an exterior domain Ω. We prove a blow-up result which generalizes the result of non-existence of global solutions of Ogawa and Takeda (2009). We also show that the critical exponent belongs to the blow-up case.

1. Introduction

This paper concerns the initial boundary value problem of the nonlinear damped wave equation in an exterior domain. Let $\Omega \subset \mathbb{R}^n$ be an exterior domain whose obstacle $\mathcal{O} \subset \mathbb{R}^n$ is bounded with smooth compact boundary $\partial\Omega$. We consider the initial boundary value problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = |u|^p & t > 0, x \in \Omega, \\ u(0,x) = u_0(x), \qquad \partial_t u(0,x) = u_1(x) & x \in \Omega, \\ u = 0, & t > 0,\ x \in \partial\Omega, \end{cases} \tag{1.1}$$

where the unknown function u is real-valued, $n \geq 1$ and $p > 1$. Throughout this paper, we assume that

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \tag{1.2}$$

and

$$\mathrm{supp} u_0 \cup \ \mathrm{supp} u_1 \text{ is compact in } \Omega. \tag{1.3}$$

To begin with, let us make mention the Cauchy problem in $\mathbb{R}^n$. Matsumura [1] showed that the existence of the global solutions of (1.1), under the assumption $n = 1, p > 3$ and $n = 2, p > 2$ for the small data. Kawashima–Nakao–Ono [2] proved the existence of a global solution with small initial data in the case $1 + 4/n \leq p < 1 + 4/(n-2)$ (see also Nakao–Ono [3]). Todorova–Yordanov [4] showed the exponent $p = 1 + 2/n$ is the critical one for (1.1), which classifies the global existence of solutions and finite time blow-up for the small data. The exponent $p = 1 + 2/n$ is known as the critical situation for the results of the semilinear heat equation. Namely, if $p > 1 + 2/n$, the solution exists globally in time, and if

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$1 < p \leq 1+2/n$ and the initial data is small, then the solution blows up in a finite time. The critical exponent $p = 1+2/n$ is called the Fujita exponent due to the work of Fujita [5]. For $1 < p < 1+2/n$, we refer the reader to [6,4,7,8] for various blowing up results. Li–Zhou [6] showed the finite time blow-up of the solution for the subcritical case $1 < p \leq 1+2/n$ and they obtained the upper bound of the lifespan of solutions for $n = 1, 2$. Todorova–Yordanov [4] proved that when $1 < p < 1+2/n$ and $n \geq 1$, the solution u blows up in a finite time. Zhang [7] extends the blow-up result to the critical case $p = 1+2/n$ by using an approach which is not only much shorter but also capable of proving more general blow-up results than those in [4].

Concerning our problem (1.1), Ikehata [9,10] proved that the solution exists globally when $n = 2$ for $p > 1+2/n$, and $n = 3, 4, 5$ for $1+4/(n+2) < p \leq 1+2/(n-2)$ under the assumption of the compactness of the initial data. Ono [11] showed the same result of global existence of solution without compactness of the initial data, applying the result of Dan–Shibata [12] and cut-off method. Ogawa–Takeda [13] proved the non-existence of non-negative global weak solutions of (1.1) under the assumption that $u_0 = 0$, $1 < p < 1+2/n$, and the support of the initial data is compact. However, in this case, the blow-up of the solutions remains an open question. In addition, up to our knowledge, the blow-up solutions of (1.1) in the critical case ($p = 1+2/n$) have not been treated in the literature.

Before stating our main theorem, we introduce some notations. For $R > 0$, let $B_R(x_0) := \{x \in \mathbb{R}^n;\ |x - x_0| < R\}$ and $\Omega_R := \Omega \cap B_R(0)$. We introduce the first eigenfunction φ_R of $-\Delta$ and the first eigenvalue λ_1 over Ω_R:

$$\begin{cases} -\Delta\varphi_R = \lambda_1\varphi_R & \text{in } \Omega_R, \\ \varphi_R > 0 & \text{in } \Omega_R, \\ \varphi_R = 0 & \text{in } \partial\Omega_R, \\ \|\varphi_R\|_{L^\infty(\Omega_R)} = 1. \end{cases} \tag{1.4}$$

The aim of this paper is to generalize the results of Ogawa–Takeda [13] by proving the blow-up of solutions of (1.1) under weaker assumptions on the initial data. Moreover, we extend this result to the critical case $p = 1+2/n$. Our results are based on the test function method introduced by Zhang [7] and modified by Fino–Karch [14]. Namely

Theorem 1.1. *Let $n \geq 1$, $1 < p \leq 1+2/n$, and $supp u_0 \cup supp u_1$ is compact. Assume, for arbitrary $0 < \epsilon_0 < 1$ and for large $R > 0$, that the initial data satisfy*

$$supp u_0 \cup supp u_1 \subset \{x \in \Omega_R;\ \varphi_R > \epsilon_0\} \tag{1.5}$$

and

$$\int_\Omega u_0(x)dx \geq 0, \qquad \int_\Omega u_1(x)\,dx > 0. \tag{1.6}$$

Then the mild solution of the problem (1.1) blows up in finite time.

This paper is organized as follows: in Section 2, we present some definitions and properties that are useful in the sequel. Section 3 is devoted to the proof of the blow-up result (Theorem 1.1).

2. Preliminaries

In this section, we introduce some notations and basic results. Using Lemma 3 in [13], there exist $C_1, C_2 > 0$ independent of R such that

$$C_1R^{-2} \leq \lambda_1 \leq C_2R^{-2}, \tag{2.1}$$

where λ_1 is the first eigenvalue of φ_R introduced in Section 1.

Now, we consider the homogeneous equation corresponding to (1.1)

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = 0 & t > 0, x \in \Omega, \\ u(0,x) = 0, \quad \partial_t u(0,x) = g(x) & x \in \Omega, \\ u = 0, & t > 0,\ x \in \partial\Omega. \end{cases}$$

By Proposition 8.4.2 in [15], this problem has a unique global solution denoted by $S(t)g$. We now give the definition of mild solutions.

Definition 2.1 (*Mild Solution*). Let $T > 0$. We say that u is a mild solution if $u \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega))$ and satisfies

$$u(t) = \partial_t S(t)u_0 + S(t)(u_0 + u_1) + \int_0^t S(t-s)|u|^p(s)ds. \tag{2.2}$$

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