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A semilinear parabolic problem with variable reaction on a general domain

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ABSTRACT

We are concerned with the parabolic equation $u_t - \Delta u = f(t)u^{p(x)}$ in $\Omega \times (0, T)$ with homogeneous Dirichlet boundary condition, $p \in C(\Omega)$, $f \in C([0, \infty))$ and Ω is either a bounded or an unbounded domain. The initial data is considered in the space $\{u_0 \in C_0(\Omega);\ u_0 \geq 0\}$. We find conditions that guarantee the global existence and the blow up in finite time of nonnegative solutions. These conditions are given in terms of the asymptotic behavior of the solution of the homogeneous linear problem $u_t - \Delta u = 0$.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a domain(bounded or unbounded) with smooth boundary $\partial\Omega$. We consider the parabolic problem

$$\begin{cases} u_t - \Delta u = f(t)u^{p(x)} & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $p \in C(\Omega)$ is a bounded function such that

$$0 < p^- \leq p(x) \leq p^+, \tag{2}$$

for all $x \in \Omega$, $p^- = \inf_{x\in\Omega} p(x)$, $p^+ = \sup_{x\in\Omega} p(x)$, $f \in C([0, \infty))$ and $u_0 \in C_0(\Omega)$.

Problem (1) occurs in many mathematical models of applied science, such as chemical reactions, heat transfer, population dynamics, electro-rheological fluids, etc. The interested readers may refer to [1–6] and the references therein. When p is a constant, there are many results about existence, uniqueness, blow up, global existence and other properties of the solution for problem (1), see for instance [7,8], and the survey [9]. When p is a function, the blow up for positive solutions of (1) was obtained in [3]. In [10,5,6] the blow up is established for solutions with positive initial energy. In [11], applying Kaplan's method and the sub and supersolutions method, blow up and global existence results for problem (1) were obtained for $\Omega = \mathbb{R}^N$ and $f \equiv 1$. More precisely, it was shown that if $p^- > 1 + 2/N$, then problem (1) possesses global nontrivial solutions. If $1 < p^- < p^+ \leq 1 + 2/N$, then all solutions of problem (1) blow up in finite time. If $p^- < 1 + 2/N < p^+$, then there are functions $p(x)$ such that problem (1) possesses global nontrivial solutions and functions $p(x)$ such that all solutions blow up. When p is constant, the value $p_F = 1 + 2/N$ is called the critical Fujita exponent for problem (1).

Considering p constant, Meier showed the following blow up and global existence result.

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Theorem 1.1 *([12]). Assume that $p(x) \equiv p$ is a constant and $f \in C([0,\infty))$.*

(i) *If $\limsup_{t\to\infty} \|S(t)u_0\|_\infty^{p-1} \int_0^t f(\sigma)d\sigma = \infty$, for all $u_0 \in C_0(\Omega)$, then every nontrivial solution of problem (1) blows up in finite time.*

(ii) *If there exists $u_0 \in C_0(\Omega)$ such that $\int_0^\infty f(\sigma)\|S(\sigma)u_0\|_\infty^{p-1}d\sigma < \infty$, then there are global positive solutions for problem (1).*

Note that Meier's result is based only on the asymptotic behavior of $\|S(t)u_0\|_\infty$ which depends on the geometry of Ω and u_0. Indeed, if Ω is a bounded domain, $\lambda_1 > 0$ is the first eigenvalue of the Laplacian operator associated to the first eigenfunction φ_1 in $H_0^1(\Omega)$ and $u_0 \sim \varphi_1$ i.e. there exist constants $C_0, C_1 > 0$ so that $C_0\varphi_1 \le u_0 \le C_1\varphi_1$, then it is well known that $\|S(t)u_0\|_\infty \sim e^{-\lambda_1 t}$ for $t > 0$. Now, if $\Omega = \mathbb{R}^N$ and $u_0 \sim |x|^{-\alpha}$ $(0 < \alpha < N)$ for $|x|$ sufficiently large, i.e. there exist constants $C_0, C_1 > 0$ so that $C_0|x|^{-\alpha} \le u_0(x) \le C_1|x|^{-\alpha}$ for $|x|$ sufficiently large, we have from [13] that $\|S(t)\|_\infty \sim t^{-\alpha/2}$.

It is important to point out two facts about Meier's result. Firstly, the result holds for any domain(bounded or unbounded). Secondly, it is sufficient to determine the value of $\|S(t)u_0\|_\infty$ to decide if a solution of problem (1) is global or nonglobal.

Theorem 1.1 has been extended to the heat equation with more general nonlinearities than s^p, in [14,15], and for some coupled parabolic systems in [16,17].

Local existence for problem (1) can be shown using the same ideas of [18,11,3]. Uniqueness may fail due to the fact that p can satisfy $p < 1$ in some subdomain of Ω. Nevertheless, we have a maximal solution $u \in C([0, T_{max}), C_0(\Omega))$ defined on a maximal interval $[0, T_{max})$ verifying

$$u(t) = S(t)u_0 + \int_0^t S(t-\sigma)f(\sigma)u(\sigma)^{p(x)}d\sigma,$$

for every $t \in [0, T_{max})$. Moreover, either $T_{\max} = \infty$ (global solution) or $T_{\max} < \infty$ and $\limsup_{t\to T_{\max}} \|u(t)\|_\infty = \infty$ (blow up solution). Henceforth, $(S(t))_{t\ge 0}$ denotes the heat semigroup with Dirichlet condition on the boundary.

The main result of this paper is to extend Meier's result to the semilinear parabolic problem (1).

Theorem 1.2. *Suppose $p \in C(\Omega)$ and $f \in C([0,\infty))$.*

(i) *Assume that $p^+ > 1$ and*

$$\limsup_{t\to\infty} \|S(t)u_0\|_\infty^{p^+-1} \int_0^t f(\sigma)d\sigma = \infty \tag{3}$$

for every $u_0 \in C_0(\Omega)$. Then every nontrivial solution of problem (1) either blow up in finite time or in infinite time. In the last case, we mean that the solution is global and $\limsup_{t\to\infty} \|u(t)\|_\infty = \infty$.

(ii) *Assume that $p^- > 1$ and there exists $w_0 \in C_0(\Omega)$, $w_0 \ge 0$, $w_0 \not\equiv 0$ verifying*

$$\int_0^\infty f(\sigma)\|S(\sigma)w_0\|_\infty^{p^--1}d\sigma < \infty. \tag{4}$$

Then there exists a constant $\Lambda > 0$, depending on p^+ and p^-, so that if $0 < \lambda < \Lambda$, then the solution u of (1), with initial data λw_0, is a non-trivial global solution. Moreover, there exists a constant $\gamma > 0$ such that $u(t) \le (1+\gamma)S(t)u_0$ for all $t \ge 0$.

Remark 1.3. If $0 < p^+ \le 1$, then all solutions of problem (1) are global. Indeed, the function $w(t) = (\|u_0\|_\infty + 1)\exp\left(\int_0^t f(s)ds\right)$, defined for all $t \ge 0$, is a global supersolution of problem (1), since $w_t = f(t)w \ge f(t)w^{p(x)}$. Hence, condition $p^+ > 1$ in item (i) is optimal only when the blow up in finite time occurs. In Theorems 1.4 and 1.5(i) we establish conditions so that only the blow up in finite time occurs.

The proof of Theorem 1.2 is based on estimates of the heat semigroup which are obtained by an iterative process. This approach simplifies the arguments for establishing global and non-global existence criteria and has been used successfully to treat some parabolic systems related in [16,17,14]. Since p is a function some difficulties arise. For instance, the estimate (6) depends on bounds for u in the interval $[0, T]$. This does not happen if p is constant.

From Theorem 1.2(i), we see that the solution of problem (1) always blows up, either in finite or infinite time. However, from the results in [11] we know that if $\Omega = \mathbb{R}^N$, $f = 1$ and $1 < p^- < p^+ \le p_F$, then only the blow up occurs. The same conclusion can be obtained when Ω contains a cone Ω_1 or Ω is bounded(see Theorem 1.4 for details).

We say that Ω_1 is a cone with vertex at the origin if $x = (r,\theta) \in \Omega_1$ where $r = |x|$ and $\theta \in D$ where $D \subset S^{N-1}$ is a region with smooth boundary ∂D. We denote by $\omega_1 = \omega_1(\Omega_1)$ the first Dirichlet eigenvalue for the Laplace–Beltrami operator on D and γ^+ the positive root of $\gamma(\gamma+N-2) = \omega_1$.

Theorem 1.4. *Assume that $p \in C(\Omega)$. Suppose that some of the following conditions hold.*

(i) *$\Omega \supset \Omega_1$, where Ω_1 is a cone with vertex at the origin, $f \in C([0,\infty))$ is such that $f \ge C_1t^q (q > -1)$ for $t \ge t_0 > 0$ and some constant $C_1 > 0$, and $1 < p^- < p^+ < p_F = 1 + 2/(N+\gamma_+)$.*

(ii) *Ω is bounded, $f(t) = e^{\beta t} (\beta > 0)$ and $1 < p^- < p^+ < p_F = 1 + \beta/\lambda_1$.*

Then, every solution of problem (1) blows up in finite time.

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