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Global well-posedness and regularity criteria for epitaxial growth models

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ABSTRACT

In this paper, we consider an epitaxial growth model with slope selection and a generalized model. First, we establish some regularity criteria of strong solutions for the epitaxial growth model with slope selection. Then, we prove the global-in-time existence of smooth solutions for a generalized model.

1. Introduction

First we consider the following epitaxial growth model with slope selection:

$$\partial_t h = -\nabla \cdot (1 - |\nabla h|^2)\nabla h - \Delta^2 h, \tag{1.1}$$

$$h(\cdot, 0) = h_0 \quad \text{in } \mathbb{R}^d, \tag{1.2}$$

where h is a scaled height function of a thin film in a co-moving frame. The fourth-order term models surface diffusion, and the nonlinear second-order term models the Ehrlich–Schwoebel effect [1–3].

The epitaxial growth of nanoscale thin films has recently received increasing interest in materials science. A major reason for this interest is that compositions like $YBa_2Cu_3O_{7-\delta}$ (YBCO) are expected to be high-temperature super-conducting and could be used in the design of semi-conductors.

Li and Liu [4] proved the global well-posedness of the problem (1.1) and (1.2) with $h_0 \in H^2$ and $d \leq 3$. The numerical analysis has been studied in [5,6]. Very recently, Li, Qiao and Tang [7] prove the global well-posedness when $h_0 \in H^{\frac{d}{2}}$ and $d \leq 3$. To know more about the growth model (1.1), we also consider the case $d \geq 4$. Fan and Zhou [8] prove the global well-posedness when $h_0 \in H^4$ and $d = 4$. As far as we know, it is completely open whether the solution exists globally or not for $d \geq 5$. The aim of this paper is to establish some regularity criteria for this case. We will prove

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Theorem 1.1. *Let $h_0 \in H^s$ with $s \geq 5$. Let h be a local strong solution to the problem (1.1) and (1.2). If one of the following two conditions*

$$\text{(i) } d = 5, 6, 7 \quad \textit{and} \quad \nabla^2 h \in C([0, T]; L^{\frac{d}{2}}), \tag{1.3}$$

$$\text{(ii) } d \geq 5, \quad \nabla h \in L^p(0, T; L^q) \textit{ with } \frac{4}{p} + \frac{d}{q} = 1 \textit{ and } d < q \leq \infty, \tag{1.4}$$

holds true for $0 < T < \infty$, then the solution h can be extended beyond $T > 0$.

Remark 1.1. If the linear second-order term-Δh is neglected in (1.1), then the system has scaling invariant property with $h \to h_\lambda := h(\lambda x, \lambda^4 t)$ for any $\lambda > 0$, thus (1.3) and (1.4) are optimal in this sense.

In [7, Remark 1.4], it was also suggested to study the following model

$$\partial_t h = -\nabla \cdot (1 - |\nabla h|^2)\nabla h - (-\Delta)^{\frac{\gamma}{2}} h \tag{1.5}$$

with $\gamma > 2$. For simplicity, we then study the following model

$$\partial_t h = -\nabla \cdot (1 - |\nabla h|^{\gamma-2})\nabla h - (-\Delta)^{\frac{\gamma}{2}} h. \tag{1.6}$$

When $\gamma = 4$, (1.6) reduces to (1.1). The aim of this paper is to study the well-posedness of (1.6) and (1.2). We will prove

Theorem 1.2. *Let $\gamma = d \geq 3$ and $h_0 \in H^s$ with $s > d$, then the problem (1.6) and (1.2) has a unique global solution h satisfying*

$$h \in L^\infty(0, T; H^s) \cap L^2\left(0, T; H^{s+\frac{d}{2}}\right) \tag{1.7}$$

for any given $T > 0$.

Remark 1.2. When $\gamma > d$, our result holds true also. However, when $2 < \gamma < d$, we are unable to prove a similar result.

Remark 1.3. Our result holds also for the problem (1.5) and (1.2).

2. Proof of Theorem 1.1

The local well-posedness of solutions has been proved in [7], we only need to establish a priori estimates.

Testing (1.1) by h, we see that

$$\frac{1}{2}\frac{d}{dt}\int h^2 dx + \int (\Delta h)^2 dx + \int |\nabla h|^4 dx = \int |\nabla h|^2 dx$$
$$\leq \|h\|_{L^2}\|\Delta h\|_{L^2} \leq \frac{1}{4}\|\Delta h\|_{L^2}^2 + \|h\|_{L^2}^2,$$

which gives

$$\|h\|_{L^\infty(0,T;L^2)} + \|h\|_{L^2(0,T;H^2)} + \|\nabla h\|_{L^4(0,T;L^4)} \leq C. \tag{2.1}$$

Testing (1.1) by $\partial_t h$, we find that

$$\frac{1}{2}\frac{d}{dt}\int (\Delta h)^2 dx + \int (\partial_t h)^2 dx + \frac{1}{4}\frac{d}{dt}\int |\nabla h|^4 dx = \frac{1}{2}\frac{d}{dt}\int (\nabla h)^2 dx. \tag{2.2}$$

Integrating (2.2) over $(0, t)$ and using (2.1) gives

$$\frac{1}{2}\int (\Delta h)^2 dx + \frac{1}{4}\int |\nabla h|^4 dx + \int_0^t \int |\partial_t h|^2 dx ds$$
$$\leq \frac{1}{2}\int |\nabla h|^2 dx + C$$
$$\leq \frac{1}{2}\|h\|_{L^2}\|\Delta h\|_{L^2} + C$$
$$\leq \frac{1}{8}\|\Delta h\|_{L^2}^2 + C,$$

which implies

$$\|h\|_{L^\infty(0,T;H^2)} + \|\partial_t h\|_{L^2(0,T;L^2)} \leq C. \tag{2.3}$$

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