ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Isotropic and anisotropic total variation regularization in electrical impedance tomography

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ARTICLE INFO

Article history:
Received 16 September 2016
Received in revised form 3 February 2017
Accepted 4 May 2017
Available online xxxx

Keywords:
Total variation regularization
Electrical impedance tomography
Diffusive tomography
Inverse problems

ABSTRACT

This paper focuses on studying the effects of isotropic and anisotropic total variation (TV) regularization in electrical impedance tomography (EIT). A characteristic difference between these two widely used TV regularization methods is that the isotropic TV is rotationally invariant and the anisotropic TV is not. The rotational variance of the anisotropic TV is known to cause geometric distortions by favoring edge orientations that are aligned with co-ordinate axes. In many applications, such as transmission tomography problems, these distortions often play only a minor role in the overall accuracy of reconstructed images, because the measurement data is sensitive to the shapes of the edges in the imaged domain. In EIT and other diffusive image modalities, however, the data is severely less sensitive to the fine details of edges, and it is an open question how large impact the selection of the TV regularization variant has on the reconstructed images. In this work, this effect is investigated based on a set of experiments. The results demonstrate that the choice between isotropic and anisotropic TV regularization indeed has a significant impact on the properties of EIT reconstructions; especially, the tendency of the anisotropic TV to favor edges aligned with co-ordinate axes is shown to yield large geometric distortions in EIT reconstructions.

1. Introduction

Electrical Impedance Tomography (EIT) produces an estimate of spatially distributed electrical conductivity inside a body based on electrical current and voltage measurements on the boundary of the body. In medicine, EIT has applications such as monitoring of lung function [1], imaging of the brain [2,3] and detection and classification of breast cancer tumors [4]. Non-medical applications of EIT include, for example, monitoring and control of industrial processes [5], non-destructive testing and monitoring [6–8].

The estimation of the spatially distributed conductivity based on current and voltage measurements amounts to estimating a coefficient of a diffusion type PDE based on the boundary data. This inverse boundary value problem is very unstable with respect to measurement and modeling errors, implying that highly accurate forward model and regularization are needed for the solution of the problem in a practical setup of working with a limited number of noisy measurements.

The regularization of the EIT problem is typically carried out by using a regularized output least squares formulation

$$\hat{\sigma} = \arg\min_{\sigma>0}\{\|V - U(\sigma)\|^2_{\Gamma_e^{-1}} + \alpha P(\sigma)\}$$

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http://dx.doi.org/10.1016/j.camwa.2017.05.004

where $P(\sigma)$ is a regularization functional, Γ_e is the covariance matrix of the measurement errors, $U(\sigma)$ is the forward map which maps σ to the electrode measurements and V is vector of voltage measurements. Traditionally, the most usual choice for the regularization functional $P(\sigma)$ has been a L_2 functional which promotes smoothness of the solution. However, in many applications of EIT the solution is expected to be piecewise regular with sharp, well-defined edges. Such applications include, for example, geophysical imaging [9], process imaging of multiphase flows [10], non-destructive testing [8], breast cancer detection [11] and stroke classification [12], to name a few. In such cases, the total variation (TV) regularization has been demonstrated as a very well-suited regularization model for EIT [9,11,13–18].

TV regularization was originally proposed for image denoising in the seminal paper [19] and it has since then been found very useful in a wide spectrum of imaging problems, see [20] for a recent review. TV regularization can be interpreted as introducing a functional which promotes sparsity of the gradient of the solution. It has basically two well-known variants, namely the *isotropic* TV

$$P(\sigma) = \int_{\Omega} \sqrt{\|\nabla\sigma\|^2 + \beta}\,\mathrm{d}x \tag{1}$$

and the *anisotropic* TV

$$P(\sigma) = \sum_k \int_{\Omega} \sqrt{\left(\frac{\partial\sigma}{\partial x_k}\right)^2 + \beta}\,\mathrm{d}x \tag{2}$$

where $\beta > 0$ is a smoothing parameter which is used to approximate the absolute value $|t| = \sqrt{t^2}$ by differentiable approximation $|t|_\beta = \sqrt{t^2+\beta}$ so that the functional $P(\sigma)$ becomes differentiable, allowing the solution of the minimization problem by classical gradient based optimization methods.

We note that there are nowadays a variety of optimization methods which allow minimization of the TV regularized inverse problem without the smoothing (i.e., using $\beta = 0$), for an application to EIT see [11]. However, we consider formulations (1) and (2), and also study how the selection of β affects the reconstructions in EIT.

It is worth noticing that referring to functional (2) as "anisotropic" is slightly misleading, because the weights corresponding to all coordinate directions are equal. However, we follow this widely spread notation introduced in [19]. For generalization of the anisotropic TV to favor different edge directions, we refer to [21]. We also note that sometimes TV regularization is implemented by considering differences of discrete values of σ across arbitrarily oriented boundaries, such as edges in a finite element mesh [11,13]. In this paper, however, we focus in the two systematic formulations (Eqs. (1) and (2)) of TV regularization.

The main difference between the properties of the two TV regularization functionals (1) and (2) is that only the isotropic TV is rotationally invariant [22]. In consequence, the use of anisotropic TV regularization can lead to geometric distortions of the images—in particular, with anisotropic TV, the boundaries in the images tend to align with the co-ordinate axes [23,24]. To illustrate this effect, Fig. 1 shows six images reconstructed based on a publicly available X-ray μCT data (90 equispaced fan beam projections from a view angle of 360°; for details of the experiment, see [25]). The first row shows the images reconstructed using the isotropic TV regularization with three regularization parameters: from left to right, $\alpha = 0.001, 0.01$ and 0.1. The higher α gets, the more impact the regularization has to the solutions, and consequently, the more details are lost from the image — indeed, with the increase of α the contrast of the image gets lower, and the boundaries within the object become smoother. The reconstructions corresponding to anisotropic TV regularization are shown in the second row. As in the case of isotropic TV, the contrast lowers in the reconstructed images as α increases. Further, as expected, the main difference between reconstructions with the isotropic and anisotropic TV with each value of α is that in the latter case, the boundaries within the imaged object are more aligned with the co-ordinate axes. With the smallest value of α, this difference is minor, yet observable with a careful examination of the images. As α increases, the effect becomes clearer.

In the example case of Fig. 1, the reconstructed images corresponding to the two TV regularization schemes differ from each other significantly only when α is too large, i.e., when the solution of the inverse problem is obviously over-regularized. Generally, in X-ray CT and other transmission tomography problems, the effect of choosing between isotropic and anisotropic TV regularization is often quite small, because in these modalities the measurement data is informative to the fine details of the edges within the target image. Indeed, the analysis in Appendix B in [26] shows that in x-ray tomography the smooth TV regularization can recover at the least the same singularities that can be recovered with classical backprojection. In EIT and other diffusive imaging modalities, however, the measurements carry less information on the details of edge shapes and this combined with the high instability of the problem suggests that the regularization can have a stronger impact on the properties of the reconstructed images at any feasible level of regularization. In this study, the effect of choosing between the isotropic and anisotropic TV regularization in EIT is investigated numerically.

2. Electrical impedance tomography

In an EIT experiment, a set of L electrodes is attached to the boundary $\partial\Omega$ of the body $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. Electric current is injected to the body through some of the electrodes and the resulting voltages are measured. The same experiment is repeated with different subsets of electrodes for the current injection (current patterns), and the objective is to estimate the conductivity $\sigma(x)$ based on the boundary measurements.