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Virtual Element Methods for hyperbolic problems on polygonal meshes

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ABSTRACT

In the present paper we develop the Virtual Element Method for hyperbolic problems on polygonal meshes, considering the linear wave equations as our model problem. After presenting the semi-discrete scheme, we derive the convergence estimates in H^1 semi-norm and L^2 norm. Moreover we develop a theoretical analysis on the stability for the fully discrete problem by comparing the Newmark method and the Bathe method. Finally we show the practical behaviour of the proposed method through a large set of numerical tests.

1. Introduction

The **Virtual Element Method** (in short, VEM or VEMs) is a very recent technique for solving partial differential equations. VEMs were lately introduced in [1] as a generalization of the finite element method on polyhedral or polygonal meshes.

The **virtual element spaces** are similar to the usual polynomial spaces with the addition of suitable (and unknown!) non-polynomial functions. The main idea behind VEM is to define **approximated discrete bilinear forms** that are **computable** only using the **degrees of freedom**. The key of the method is to define suitable **projections** (for instance gradient projection or L^2 projection) onto the space of polynomials that are computable on the basis of the degrees of freedom. Using these projections, the bilinear forms (e.g. the stiffness matrix, the mass matrix and so on) require only integration of polynomials on the (polytopal) element in order to be computed. Moreover, the ensuing discrete solution is conforming and the accuracy granted by such discrete bilinear forms turns out to be sufficient to recover the correct order of convergence. Following such approach, VEM is able to make use of very general polygonal/polyhedral meshes without the need to integrate complex non-polynomial functions on the elements (as polygonal FEM do) and without loss of accuracy. As a consequence, VEM is not restricted to low order convergence and can be easily applied to three dimensions and use non convex (even non simply connected) elements.

An additional peculiarity of the VEMs is the satisfaction of the **patch test** used by engineers for testing the quality of the methods. Roughly speaking, a method satisfies the patch test if it is able to give the exact solution whenever this is a global polynomial of the selected degree of accuracy.

In [2] the authors introduce a variant of the virtual element method presented in [1] that allows to compute the exact L^2 projection of the virtual space onto the space of polynomials and extends the VEMs technology to the three-dimensional case. A helpful paper for the computer implementation of the method is [3]. In [4] the authors construct Virtual Element spaces that are $H(\text{div})$-conforming and $H(\text{curl})$-conforming.

The Virtual Element Method has been developed successfully for a large range of problems: the linear elasticity problems, both for the compressible and the nearly incompressible case [5,6], a stream formulation of VEMs for the Stokes problem [7],

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the non-linear elastic and inelastic deformation problems, mainly focusing on a small deformation regime [8], the Darcy problem in mixed form [9], the plate bending problem [10], the Steklov eigenvalue problem [11], the general second order elliptic problems in primal [12] and mixed form [13], the Cahn–Hilliard equation [14], the Helmholtz problem [15], the discrete fracture network simulations [16,17], the time-dependent diffusion problems [18] and the Stokes problem [19]. In [20,21] the authors present a non-conforming Virtual Element Space. Finally in [22] the authors introduce the last version of Virtual Element spaces, the Serendipity VEM spaces that, in analogy with the Serendipity FEMs, allows to reduce the number of degrees of freedom.

Recently in [23,24], Mimetic Finite Difference methods [25] (technique having common features with VEM) have been applied to the space discretization of PDEs of parabolic and hyperbolic type in two dimension, showing how this technique preserves invariants of the solution better than classical space discretizations such as finite difference methods. In the present contribution we develop the Virtual Element Method for hyperbolic problems. We consider as a model problem the classical time-dependent wave equations. The discretization of the problem requires the introduction of two discrete bilinear forms, one being the approximated grad–grad form of the stationary case [1] and the other being a discrete counterpart of the L^2 scalar product. The latter is built making use of the enhancements techniques of [2]. In the paper we focus our attention on the bi-dimensional case and we develop a full theoretical analysis, first bounding the error between the semi-discrete and the continuous problems and later giving two examples of fully discrete problems. Finally, a large range of numerical tests in accordance with the theoretical derivations is presented.

The paper is organized as follows. In Section 2 we introduce the model continuous problem. In Section 3 we present the VEM discretization and the analysis of the error for the semi-discrete problem. In Section 4 we detail the theoretical features of the fully discrete scheme, in particular we analyse the convergence and the stability properties for the fully discrete problem by using the Newmark method and the Bathe method as time integrator method. Finally, in Section 5 we show the numerical tests.

2. The continuous problems

We consider the second order evolution problem in time, in particular we study the wave equation as model hyperbolic problem. Let $\Omega \subset \mathbb{R}^2$ be the polygonal domain of interest. Then the mathematical problem is given by:

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = z_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where u represents the unknown variable of interest, u_t and u_{tt} denote respectively its first and second order time derivative. We assume the external force $f \in L^2(\Omega \times (0,T))$ and the initial data $u_0, z_0 \in H^1_0(\Omega)$. Then a standard variational formulation of Problem (1) is:

$$\begin{cases} \text{find } u \in C^0(0,T;\, H^1_0(\Omega)) \cap C^1(0,T;\, L^2(\Omega)), \text{ such that} \\ (u_{tt}(t),\, v) + a(u(t),\, v) = (f(t),\, v) \quad \text{for all } v \in H^1_0(\Omega), \text{ for a.e. } t \text{ in } (0,T) \\ u(0) = u_0, \qquad u_t(0) = z_0, \end{cases} \tag{2}$$

where

- the derivative u_{tt} above is to be intended in the weak sense in $(0,T)$,
- $(\cdot,\cdot)\colon L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ denotes the standard L^2 scalar product on Ω,
- $a(\cdot,\cdot)\colon H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ denotes the grad–grad form $a(u,\, v) = (\nabla u,\, \nabla v)$.

It is well known (see for instance [26]) that the bilinear form $a(\cdot,\cdot)$ is continuous and coercive, i.e. there exist two uniform positive constants a and α such that

$$a(u,\, v) \le a\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \qquad a(v,\, v) \ge \alpha\|v\|^2_{H^1(\Omega)} \quad \text{for all } u,\, v \in H^1_0(\Omega),$$

then Problem (2) has a unique solution $u(t)$ such that

$$\left(a(u(t),\, u(t)) + \|u_t(t)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \le \left(a(u_0,\, u_0) + \|z_0\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} + |f|_{L^1(0,t,\,L^2(\Omega))} \quad \forall t \in (0,T).$$

In the rest of the paper we will make use of the following notation. We will indicate the classical Sobolev semi-norms (and analogously for the norms) with the shorter symbols

$$|v|_s = |v|_{H^s(\Omega)}, \qquad |v|_{s,\omega} = |v|_{H^s(\omega)}$$

for any non-negative constant $s \in \mathbb{R}$, open subset $\omega \subseteq \Omega$ and for all $v \in H^s(\Omega)$, while C will denote a generic positive constant independent of the mesh diameter h and time step size τ and that may change at each occurrence.