Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The critical exponent for the dissipative plate equation with power nonlinearity

M. D'Abbicco

Department of Mathematics, University of Bari, Via E. Orabona 4, 70125 Bari, Italy

ARTICLE INFO

Article history:
Available online xxxx

Keywords:
Damped plate equation
Critical exponent
Rotational inertia
Regularity loss type decay
Power nonlinearity
System of plate equations

ABSTRACT

In this paper, we find the critical exponent of global small data solutions for a damped plate equation with power nonlinearity

$$u_{tt} - \Delta u_{tt} + \Delta^2 u + u_t = |u|^p, \qquad t \geq 0,\ x \in \mathbb{R}^2,$$

and for a system of two weakly coupled damped plate equations. We show how assuming small data in the energy space $H^2 \times H^1$ and in L^1 is sufficient to compensate the *regularity-loss* type decay effect created by the rotational inertia term $-\Delta u_{tt}$.

1. Introduction

In this paper, we prove that the critical exponent for the damped plate equation with power nonlinearity

$$\begin{cases} u_{tt} - \Delta u_{tt} + \Delta^2 u + u_t = |u|^p, & t \geq 0,\ x \in \mathbb{R}^2, \\ (u, u_t)(0, x) = (u_0, u_1)(x), \end{cases} \tag{1}$$

is 3. By critical exponent, we mean that small data solutions exist globally in time, if $p \in (3, \infty)$, whereas global solutions to (1) cannot exist, under suitable sign assumption on the data, for $p \in (1, 3]$. In this sense, for any $p > 3$, problem (1) can be considered as a small perturbation of the linear problem

$$\begin{cases} u_{tt} - \Delta u_{tt} + \Delta^2 u + u_t = 0, & t \geq 0,\ x \in \mathbb{R}^2, \\ (u, u_t)(0, x) = (u_0, u_1)(x). \end{cases} \tag{2}$$

Model (2) regulates the evolution of a transversal displacement of a thin plate occupying the domain $\mathbb{R}^2$, under the action of a frictional dissipation u_t and of rotational inertia effects $-\Delta u_{tt}$ (see [7]). We address the reader to [1] and the reference therein, for a detailed investigation of properties like existence and uniqueness of the solution to (2). In [1], the authors derive several energy estimates and L^2 estimates for the solution to (2), and they apply these results to study the semilinear problem with nonlinearities like $|u_t|^p$, with $p > 2$. They also study the problem in n space dimensions, see (50), for $n \leq 5$, whereas the general case $n \geq 1$ has been investigated for nonlinearities like u_t^2 in [8].

The energy for (2) is

$$E(t) = \frac{1}{2}\,\|u_t(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\,\|\nabla u_t(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\,\|\Delta u(t,\cdot)\|_{L^2}^2,$$

E-mail address: m.dabbicco@gmail.com.
URL: http://www.dabbicco.com.

http://dx.doi.org/10.1016/j.camwa.2017.02.045

and it dissipates, due to

$$E'(t) = -\|u_t(t,\cdot)\|_{L^2}^2,$$

so that, a natural solution space for problem (1) is $\mathcal{C}([0,\infty),H^2)\cap\mathcal{C}^1([0,\infty),H^1)$.

However, the presence of the *rotational inertia* term $-\Delta u_{tt}$ makes harder to obtain estimates for the linear problem (2), if compared with the corresponding problem without rotational inertia, or with the damped wave equation. As a consequence, it is more difficult to apply these linear estimates to study the semilinear problem for small data (the problem for the semilinear damped wave equation has been solved in [10]; see [4] for more general dissipative evolution equations).

Indeed, the rotational inertia term creates a structure of *regularity-loss* type decay in the linear problem; in general, one has

$$\|\partial_t^k\partial_x^\alpha u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\min\left\{\frac{|\alpha|+1}{4}+k,\,\frac{s}{2}\right\}}\left(\|u_0\|_{L^1}+\|u_1\|_{L^1}+\|u_0\|_{H^{|\alpha|+k+s}}+\|u_1\|_{H^{|\alpha|+k+s-1}}\right),$$

for any $t\in[0,\infty)$, for any $k\in\mathbb{N}$, $\alpha\in\mathbb{N}^2$, $s\geq 0$. In particular, there is a competition between a decay rate which increases when the space dimension increases and when one takes more derivatives, and a decay rate which increases when more data regularity is assumed. This property has been found and investigated in several models (see the references in [8]).

More precisely, on the one hand, the low-frequencies part of the solution to (2) has the same asymptotic profile of the solution to the parabolic problem

$$\begin{cases}\varphi_t+\Delta^2\varphi=0, & t\geq 0,\ x\in\mathbb{R}^2,\\ \varphi(0,x)=u_0(x)+u_1(x), \end{cases} \tag{3}$$

i.e., $\varphi = e^{-t\Delta^2}(u_0+u_1)$. This effect is called *diffusion phenomenon,* and we address the reader to [3] for a more general presentation of this phenomenon for dissipative evolution equations. On the other hand, the high-frequencies part of the solution to (2) has a behavior “like”

$$e^{-\frac{t}{2}(1-\Delta)^{-1}}w(t,x)=\mathfrak{F}^{-1}\left(e^{-\frac{t}{2(1+|\xi|^2)}}\hat{w}(t,\xi)\right),$$

where w is the solution to the Cauchy problem for the wave equation:

$$\begin{cases}w_{tt}-\Delta w=0, & t\geq 0,\ x\in\mathbb{R}^2,\\ (w,w_t)(0,x)=(u_0,u_0+u_1)(x). \end{cases} \tag{4}$$

Here, and in the following, we denote by $\hat{f}=\mathfrak{F}(f)$, the Fourier transform of a function f, and by $\mathfrak{F}^{-1}$ the inverse of the Fourier transform.

In this paper we show with a quite simple strategy how this *regularity-loss* type decay can be compensated to study (1), by choosing suitable energy estimates with appropriate data regularity. It is easy to show that the critical exponent for (1) is expected to be 3. The nonexistence of global solutions to (1), for $p\in(1,3]$, under a suitable sign assumption on the initial data, may be proved by using the test function method [5], as done for the damped wave equation [11].

In our main result, we show that the assumption of small data in the energy space $H^2\times H^1$ and in L^1, is sufficient to prove the global existence of the solution u to (1), for any $p>3$, and to derive optimal estimates for $\|u(t,\cdot)\|_{L^r}$, for any $r\in[2,\infty]$. By optimal estimates we mean, here and in the following, that the decay rate as $t\to\infty$ is the same as the decay rate for the solution to (3). This implies that such a decay rate cannot be improved by assuming additional data regularity.

Theorem 1. *Let $p>3$. Then there exists $\varepsilon>0$ such that for any data*

$$\begin{aligned}&(u_0,u_1)\in\mathcal{A} := (L^1\cap H^2)\times(L^1\cap H^1), \qquad \textit{with}\\ &\|(u_0,u_1)\|_{\mathcal{A}} := \|u_0\|_{L^1}+\|u_0\|_{H^2}+\|u_1\|_{L^1}+\|u_1\|_{H^1}\leq\varepsilon,\end{aligned} \tag{5}$$

there exists a unique solution

$$u\in\mathcal{C}([0,\infty),H^2)\cap\mathcal{C}^1([0,\infty),H^1), \tag{6}$$

to (1). Moreover, it satisfies the following estimates

$$\|u(t,\cdot)\|_{L^r}\lesssim(1+t)^{-\frac12\left(1-\frac1r\right)}\|(u_0,u_1)\|_{\mathcal{A}}, \qquad \forall r\in[2,\infty], \tag{7}$$

$$\|(\nabla u,u_t)(t,\cdot)\|_{L^2}\lesssim(1+t)^{-\frac12}\|(u_0,u_1)\|_{\mathcal{A}}, \tag{8}$$

$$\|(\Delta u,\nabla u_t)(t,\cdot)\|_{L^2}\lesssim\|(u_0,u_1)\|_{\mathcal{A}}. \tag{9}$$

Estimate (7) is optimal, since the decay rate $(1+t)^{-\frac12\left(1-\frac1r\right)}$ is the same as the decay for the L^r norm of the solution to (3). Also estimate (8) is optimal for $\|\nabla u(t,\cdot)\|_{L^2}$, but not for $\|u_t(t,\cdot)\|_{L^2}$.

In our second result, we consider a system of two weakly coupled damped plate equations

$$\begin{cases}u_{tt}-\Delta u_{tt}+\Delta^2u+u_t=|v|^p, & t\geq 0,\ x\in\mathbb{R}^2,\\ v_{tt}-\Delta v_{tt}+\Delta^2v+v_t=|u|^q, & t\geq 0,\ x\in\mathbb{R}^2,\\ (u,u_t,v,v_t)(0,x)=(u_0,u_1,v_0,v_1)(x).\end{cases} \tag{10}$$