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Null controllability and numerical method for Crocco equation with incomplete data based on an exponential integrator and finite difference-finite element method

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ABSTRACT

We consider the linearized Crocco equation in fluid dynamics with incomplete data and Robin boundary conditions, and address theoretical and numerical distributed null control. The controllability problem is solved through a dual reformulation. We first resolve some constrained extremal problems and apply appropriated duality techniques that lead to the formulation of equivalent unconstrained extremal problem in variational form. Novel numerical technique based on finite difference-finite element space discretization and exponential integrator in time discretization is proposed. For illustration, numerical simulations are provided.

1. Introduction

It is well known that the velocity field of a laminar flow on a flat plate can be described by the Prandtl equations [1]. Prandtl equations have been of interest to many researchers (see [2] and references therein). These equations are written in an unbounded domain $(0, L) \times (0, \infty)$ for a two dimensional flow. Note that $(0, L)$ represents the part of the plate where the flow is laminar, and $(0, \infty)$ represents the thickness of the boundary layer. Usually, the matching conditions with the external flow are stated at ∞. However using the so-called Crocco transformation, these equations are transformed to a nonlinear degenerate parabolic equation (see [1,2]) called Crocco equations in a bounded domain $\Omega = (0, L) \times (0, 1)$ more feasible for numerical approximations. Let $T > 0$ be the final time, in this work, we consider the following linearized Crocco equation around a stationary solution with incomplete parameters

$$\begin{cases} \dfrac{\partial u}{\partial t} + d(x)\dfrac{\partial u}{\partial s} - \dfrac{\partial^2 u}{\partial x^2} = 0 & (t,s,x) \in Q = (0,T)\times\Omega, \\ u(t,s,0) = 0 & (t,s) \in (0,T)\times(0,L), \\ \dfrac{\partial u}{\partial x}(t,s,1) = \xi + \lambda\hat{\xi} & (t,s) \in (0,T)\times(0,L), \\ u(t,0,x) = u_1 & (t,x) \in (0,T)\times(0,1), \\ u(0,s,x) = u^0 + \tau\hat{u}^0 & (s,x) \in \Omega \end{cases} \tag{1}$$

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where u_1 depends on the incident velocity of the flow, and d is a regular coefficient. Note that t is the time and (s, x) are spatial variables in the scaled domain Ω.

The boundary condition $\xi + \lambda\hat{\xi}$ depends on the viscosity forces and is partially known. The initial datum $u^0 + \tau\hat{u}^0$ is also partially known. Furthermore ξ and u^0 are given in $L^2((0,T) \times (0,L))$ and $L^2(\Omega)$ respectively. The data of the system (1) are incomplete in the following sense:

- $\hat{u}^0 \in L^2(\Omega)$ is unknown but $\left\|\hat{u}^0\right\|_{L^2(\Omega)} = 1$,
- $\hat{\xi} \in L^2((0,T) \times (0,L))$ is unknown but $\left\|\hat{\xi}\right\|_{L^2((0,T)\times(0,L))} = 1$,
- $\lambda \in \mathbb{R}$ is unknown but very small,
- $\tau \in \mathbb{R}$ is unknown but also very small.

We aim to obtain information on (to evaluate) the missing flux $\lambda\hat{\xi}$ that is independent of the variation of the initial data around u^0 using the instantaneous sentinel method. This term is usually called pollution term. We denote by $\Theta = (a, b) \times (a_1, b_1)$ the observation domain, $\omega = (\alpha, \beta) \times (\alpha_1, \beta_1)$ the domain of the control and u_{obs} the observation of the system. To proceed, for a given $h \in L^2(\Theta)$ and $v \in L^2(\omega)$ called the control term, we define the functional S applied at u, the solution of the system (1) by

$$S(u) = \int_{\omega} vu(T, s, x)\, dxds + \int_{\Theta} hu(T, s, x)\, dxds. \tag{2}$$

The functional S is said to be the instantaneous sentinel if it exists $v \in L^2(\omega)$ of minimum norm such that $\frac{\partial S(u)}{\partial \tau} = 0$. The evaluation of the pollution term by the instantaneous sentinel method is done in two steps. In the first step called auxiliary part, we determine the control v (existence and characterization) which ensures the existence of the instantaneous sentinel. The last step establishes from the instantaneous sentinel an integral equation where the unknown is the pollution term.

The sentinel method was introduced in [3] as a particular least squares method. Many papers (see [4,5] and references therein) have used the definition in [3] in both theoretical and numerical aspects. In [6], the sentinel method and the instantaneous method are used to calculate the pollution term on a semilinear diffusion–reaction equation. In [1,7], the instantaneous sentinel based on the Hilbert uniqueness method is used to study the pollution term in incomplete data problems. In [8], the penalization method is used to prove a null controllability of a nonlinear dissipative system with application to the sentinel. Although the Hilbert uniqueness method is well used, this method is not feasible for numerical approximation of the continuous problem, despite the fact that the existence and uniqueness are ensured.

In this work, we use the penalization method for the controllability problem and the instantaneous sentinel to evaluate the unknown flow in a laminar flow problem. This double reformulation based on the penalization method and the duality theory leads to a variational formulation, more feasible for numerical approximation of the sentinel. Further, novel numerical technique based on finite difference- finite element for space discretization and exponential integrator in time integration is proposed to approximate the control term v, which can therefore be used to have more information about the pollution term. For illustration, some numerical simulations of v are also provided.

The paper is divided as follows. In Section 2, we provide some theoretical results along with their proofs. The dual reformulation is used in Section 3 to obtain the variational form of the control term, and the integral equation of the pollution term $\lambda\hat{\xi}$ is derived. Our novel numerical technique to approximate the control term along with numerical simulations is provided in Section 4.

2. Well posedness and main result

Let $C_b\left(0, L; L^2\big((0,T) \times (0,1)\big)\right)$ be the space of bounded and continuous functions from $(0, L)$ into $L^2\big((0,T) \times (0,1)\big)$ and $C^0\big((0,T); L^2(\Omega)\big)$ the space of continuous functions from $[0, T]$ into $L^2(\Omega)$.

For given data parameters of the problem (1), we have the following well posedness result.

Proposition 2.1. *Assume that $d \in L^\infty(0,1)$ and $d(x) > 0$ for every $x \in [0,1]$, then the problem (1) admits a unique solution u with the following regularity*

$$u \in C^0\left((0,T); L^2(\Omega)\right) \cap L^\infty\left((0,L); L^2\big((0,T)\times(0,1)\big)\right) \cap L^2\left((0,T)\times(0,L); H_0^1(0,1)\right).$$

Moreover

$$\sqrt{d}u \in C_b\left(0, L; L^2\big((0,T)\times(0,1)\big)\right),$$