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On the virtual element method for topology optimization on polygonal meshes: A numerical study[☆]

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ABSTRACT

It is well known that the solution of topology optimization problems may be affected both by the geometric properties of the computational mesh, which can steer the minimization process towards local (and non-physical) minima, and by the accuracy of the method employed to discretize the underlying differential problem, which may not be able to correctly capture the physics of the problem. In light of the above remarks, in this paper we consider polygonal meshes and employ the virtual element method (VEM) to solve two classes of paradigmatic topology optimization problems, one governed by nearly-incompressible and compressible linear elasticity and the other by Stokes equations. Several numerical results show the virtues of our polygonal VEM based approach with respect to more standard methods.

1. Introduction

The study of numerical methods for the approximation of partial differential equations on polygonal and polyhedral meshes is drawing the attention of an increasing number of researchers (see, e.g., the special issues [1,2] for a recent overview of the different methodologies). Among the different proposed methodologies, here we focus on the Virtual Element Method (VEM) which has been introduced in the pioneering paper [3] and can be seen as an evolution of the Mimetic Finite Difference method, see, e.g., [4,5] for a detailed description. Recently, VEM has been analyzed for general elliptic problems [6,7], linear and nonlinear elasticity [8–10], plate bending [11,12], Cahn–Hilliard [13], Stokes [14,15], Helmholtz [16], parabolic [17], Steklov eigenvalue [18], elliptic eigenvalue [19] and discrete fracture networks [20]. In parallel, several different variants of the VEM have been proposed and analyzed: mixed [21,22], discontinuous [23], $H(\mathrm{div})$ and $H(\mathbf{curl})$-conforming [24], hp [25], serendipity [26] and nonconforming [27–31] VEM.

Such a flourishing research activity founds an important motivation in the great flexibility that the use of polytopal meshes can ensure in dealing with problems posed on very complicated and possibly deformable geometries. In this respect, as first recognized by G.H. Paulino and his collaborators in a series of ground breaking papers [32–36], topology optimization represents an intriguing challenge for the use of polyhedral meshes. Topology optimization is a fertile area of research

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that is mainly concerned with the automatic generation of optimal layouts to solve design problems in Engineering. The classical formulation addresses the problem of finding the best distribution of an isotropic material that minimizes the work of the external loads at equilibrium, while respecting a constraint on the assigned amount of volume. This is the so-called minimum compliance formulation that can be conveniently employed to achieve stiff truss-like layout within a two-dimensional domain. A classical implementation resorts to the adoption of four node (displacement-based) finite elements that are coupled with an elementwise discretization of the (unknown) density field. When regular meshes made of square elements are used, well-known numerical instabilities arise, see in particular the so-called checkerboard patterns. On the other hand, when unstructured meshes are needed to cope with complex geometries, additional instabilities can steer the optimizer towards local minima instead of the expected global one. Unstructured meshes approximate the strain energy of truss-like members with an accuracy that is strictly related to the geometrical features of the discretization, thus remarkably affecting the achieved layouts. On this latter issue, as pointed out also in [32], the use of polyhedral meshes provides flexibility in the difficult computational task of meshing complex geometries, while, in parallel, it can contribute to avoid that the geometry of the mesh dictates the possible layout of material and the orientation of members, thus excluding physical optimal configurations from the final design obtained by the numerical procedure.

The aim of this paper is to push forward the study of [32,35,36]. In [32] the authors analyze the possibility of avoiding sub-optimal (non-physical) layout in topology optimization for structural applications when *polygonal finite elements* and polytopal meshes are employed, whereas in [36] the virtual element method is employed for solving compliance minimization and compliant mechanism problems in three dimensions. In [35] *polygonal finite elements* are employed to solve topology optimization problems governed by Stokes equations on polygonal meshes. In view of the above contributions, and with the goal of deepening the comprehension of the role of VEM and polygonal meshes in topology optimization, we focus on the use of this latter method for solving topology optimization governed by linear elasticity (compressible and nearly-incompressible) and Stokes flow. For each of the above examples, we systematically consider the impact that the combined approach VEM and polygonal meshes has on the quality of the obtained layout and compare them with the ones provided by standard approaches.

The outline of the paper is the following. In Section 2 we present the continuous formulation of the topology optimization problems that we will consider throughout the paper, while in Section 3 we introduce the corresponding virtual element discretizations. In Section 4 we present and extensively discuss several numerical experiments assessing the virtues of the combined use of VEM and polygonal meshes in solving each of the previously introduced topology optimization problems. Finally, in Section 5 we draw some conclusion.

2. Topology optimization problems: continuous formulation

We briefly recall the continuous formulations of the topology optimization problems we are interested in, namely the minimum compliance problem governed by the linear elasticity equation (Section 2.1) and the optimal flow problem governed by the Stokes equation (Section 2.2). We first recall some notation that will be useful in the following. Let Ω be a two-dimensional bounded, polygonal domain with boundary $\Gamma = \partial\Omega$ and let $\Gamma_d \subset \Gamma$ be a subset of the boundary of the domain. We introduce the following spaces

$$\mathcal{V}_0 = \{\mathbf{u} \in (H^1(\Omega))^2 : \ \mathbf{u} = \mathbf{0} \text{ on } \Gamma_d\}$$
$$\mathcal{V}_d = \{\mathbf{u} \in (H^1(\Omega))^2 : \ \mathbf{u} = \mathbf{u_d} \text{ on } \Gamma_d\}$$

where $\mathbf{u_d}$ is a possibly null given function. Moreover, let us introduce the control space

$$\mathcal{Q}_{\rm ad} = \{\rho \in L^\infty(\Omega) : \ 0 < \rho_{\min} \le \rho \le 1 \text{ a.e. in } \Omega\}$$

of bounded functions representing the material density in Ω, where $\rho_{\min}$ is some positive lower bound.

2.1. Minimum compliance

In this section we shortly describe the topology optimization problem for minimum compliance. This corresponds to find the optimal distribution of a given amount of linear elastic isotropic material (described by an element of $\mathcal{Q}_{\rm ad}$) such that the work of the external load against the corresponding displacement at equilibrium is minimized.

More precisely, let λ_0 and μ_0 be the Lamé coefficients of the given material and introduce the bilinear form $a(\cdot,\cdot) : (H^1(\Omega))^2 \times (H^1(\Omega))^2 \to \mathbb{R}$ defined as follows

$$a(\mathbf{u},\mathbf{v}) = 2\mu_0 \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\, d\mathbf{x} + \lambda_0 \int_\Omega \operatorname{div}\mathbf{u}\, \operatorname{div}\mathbf{v}\, d\mathbf{x}, \qquad (1)$$

where $\epsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla^T\mathbf{u}\right)$ is the symmetric gradient. Moreover, let us introduce the semi-linear form $a(\rho;\cdot,\cdot) : \mathcal{Q}_{\rm ad} \times (H^1(\Omega))^2 \times (H^1(\Omega))^2 \to \mathbb{R}$

$$a(\rho;\mathbf{u},\mathbf{v}) = 2\int_\Omega \mu(\rho)\epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\, d\mathbf{x} + \int_\Omega \lambda(\rho)\operatorname{div}\mathbf{u}\, \operatorname{div}\mathbf{v}\, d\mathbf{x} \qquad (2)$$

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