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Lower bounds for the blow-up time in a superlinear hyperbolic equation with linear damping term

Khadijeh Baghaei

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran

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ABSTRACT

This paper is concerned with the blow-up of solutions to a superlinear hyperbolic equation with linear damping term

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = |u|^{p-2}u, \qquad \text{in } [0, T] \times \Omega,$$

where $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, is a bounded domain with smooth boundary and $T > 0$. Here, $\omega \geq 0$ and $\mu > -\omega\lambda_1$, where λ_1 is the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions. The recent results show that in the case of $\omega > 0$, if $2 < p < \infty$ $(n = 1, 2)$ or $2 < p \leq \frac{2n}{n-2}$ $(n \geq 3)$, then the solutions to the above equation blow up in a finite time under some suitable conditions on initial data.

In this paper, in the case of $\omega > 0$, we obtain a lower bound for the blow-up time when the blow-up occurs. This result extends the recent results obtained by Sun et al. (2014) and Guo and Liu (2016).

1. Introduction

In this paper, we consider the following superlinear hyperbolic equation with linear damping term:

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = |u|^{p-2}u, & \text{in } [0, T] \times \Omega, \\ u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x), & \text{in } \Omega, \\ u(x, t) = 0, & \text{on } [0, T] \times \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, is a bounded domain with smooth boundary, $T > 0$, $\omega \geq 0$ and $\mu > -\omega\lambda_1$, where λ_1 is the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions. Here, $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ are the initial value functions. Also, p satisfies the following condition:

$$2 < p \leq \begin{cases} \dfrac{2n}{n-2}, & \text{for } \omega > 0, \\ \dfrac{2n-2}{n-2}, & \text{for } \omega = 0. \end{cases} \qquad n \geq 3; \ \text{or } 2 < p < \infty, \ n = 1, 2. \tag{1.2}$$

Problems like (1.1) can be used as models of viscoelastic fluids, processes of filtration through a porous media and fluids with temperature-dependent viscosity. There are much more works related to damped wave equations in the literature, we refer the interested readers to [1–12] and the references therein.

E-mail addresses: khbaghaei@ipm.ir, kh.baghaei@gmail.com.

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Gazzola and Squassina [13] studied problem (1.1). They proved that problem (1.1) has a unique local solution which is global provided that $p \in (1, 2]$, whereas for $p > 2$ under the condition (1.2), the solution of (1.1) blows up in a finite time. More precisely, under the condition (1.2), if u be the unique solution to (1.1), then $T_{\max} < \infty$ if and only if there exists $\bar{t} \in [0, T_{\max})$ such that:

$$u(\bar{t}) \in \left\{ u \in H_0^1(\Omega) : \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{p}\|u(t)\|_p^p \leq d \text{ and } \|\nabla u(t)\|_2^2 - \|u(t)\|_p^p < 0 \right\}$$

and

$$\frac{1}{2}\|\nabla u(\bar{t})\|_2^2 + \frac{1}{2}\|u_t(\bar{t})\|_2^2 - \frac{1}{p}\|u(\bar{t})\|_p^p \leq d,$$

where $T_{\max}$ denotes the maximal existence time and $d = \frac{p-2}{2p}\left(\inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla u(t)\|_2^2}{\|u(t)\|_p^2}\right)^{\frac{p}{p-2}}$. Besides, they showed that if $T_{\max} < \infty$, then

$$\lim_{t \to T_{\max}^-} \|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 = \infty \quad \text{and} \quad \lim_{t \to T_{\max}^-} \|u(t)\|_p = \infty. \tag{1.3}$$

Note that if $T_{\max} < \infty$, then the solution of (1.1) blows up and $T_{\max}$ is the blow up time.

For $n \geq 3$, Sun et al. [14] obtained a lower bound for the blow-up time when $2 < p \leq \frac{2(n-1)}{n-2}$. Recently, Guo and Liu [15] improved the restriction p and for $\frac{2(n-1)}{n-2} < p \leq \frac{2(n^2-2)}{n(n-2)}$ found a lower bound for the blow-up time.

In view of (1.2), we see that in the case of $\omega = 0$ and $n \geq 3$, the results of [14] are sufficient. Also, their method can be used for $\omega = 0$ and $n = 1, 2$. Thus, in the present paper, we will study problem (1.1) with $\omega > 0$ under the condition (1.2). We will obtain a lower bound for the blow-up time when $2 < p < \infty$ $(n = 1, 2)$ or $2 < p \leq \frac{2n}{n-2}$ $(n \geq 3)$. This result extends the recent results obtained in [14,15]. In the next section, we will prove our main result.

2. The main result

Here, we have our main results.

Theorem 2.1. *Suppose that $\omega > 0$ and $\mu > -\omega\lambda_1$, where λ_1 is the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions and $2 < p < \infty$ $(n = 1, 2)$ or $2 < p \leq \frac{2n}{n-2}$ $(n \geq 3)$. Also, suppose that $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $\int_\Omega u_0 u_1 \, dx > 0$ and:*

$$\frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{p}\| u_0\|_p^p \leq d, \qquad \|\nabla u_0\|_2^2 - \| u_0\|_p^p < 0, \qquad \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\| u_1\|_2^2 - \frac{1}{p}\| u_0\|_p^p \leq d. \tag{2.1}$$

Then the solution of (1.1) blows up in a finite time and the following estimate holds:

$$T_{\max} \geq \int_{\varphi(0)}^{+\infty} \frac{d\xi}{c_2 \xi^{\frac{\alpha(p-1)}{p(\alpha-1)}} + c_3}, \tag{2.2}$$

where $1 < \alpha < 2$ and c_2, c_3 are some positive constants to be determined later in the proof and:

$$\varphi(0) = \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\|u_1\|_2^2 + \frac{1}{p}\|u_0\|_p^p. \tag{2.3}$$

Proof. By considering the results of Gazzola and Squassina [13], under the conditions (1.2) and (2.1), the solution of (1.1) blows up in a finite time. Thus, we obtain a lower bound for the blow-up time. Define

$$\varphi(t) = \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{p}\|u(t)\|_p^p.$$

Then,

$$\varphi'(t) = \int_\Omega \nabla u(t) \cdot \nabla u_t(t) \, dx + \int_\Omega u_t(t) u_{tt}(t) \, dx + \int_\Omega |u(t)|^{p-2} u(t) u_t(t) \, dx. \tag{2.4}$$

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