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On condition numbers for least squares with quadric inequality constraint

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ABSTRACT

In this paper, we will study normwise, mixed and componentwise condition numbers for the linear mapping of the solution for general least squares with quadric inequality constraint (GLSQI) and its standard form (LSQI). We will introduce the mappings from the data space to the interested data space, and the Fréchet derivative of the introduced mapping can deduced through matrix differential techniques. Based on condition number theory, we derive the explicit expressions of normwise, mixed and componentwise condition numbers for the linear function of the solution for GLSQI and LSQI. Also, easier computable upper bounds for mixed and componentwise condition numbers are given. Numerical example shows that the mixed and componentwise condition numbers can tell us the true conditioning of the problem when its data is sparse or badly scaled. Compared with normwise condition numbers, the mixed and componentwise condition number can give sharp perturbation bounds.

1. Introduction

Least squares problems with quadratic constraints arise in a variety of applications, such as smoothing of noisy data, the solution of discretized ill-posed problems from inverse problem, and in trust region methods for nonlinear least squares problems; see the monograph [1] and references therein.

The *general least squares with quadric inequality constraint* (GLSQI) [1] is formulated by

$$\min_{x\in\mathbb{R}^n} \|Ax-b\|_2 \quad \text{subject to } \|Cx-d\|_2 \le \gamma, \tag{1.1}$$

where $A\in\mathbb{R}^{m\times n}$, $C\in\mathbb{R}^{p\times n}$, $b\in\mathbb{R}^m$, $d\in\mathbb{R}^p$ and $\gamma>0$. The existence and uniqueness of the solution to GLSQI had been investigated by Gander [2]. Let $x_{A,C}$ be the solution of the problem

$$\min_{x\in S} \|Cx-d\|_2, \quad S=\left\{x\in\mathbb{R}^n \,|\, \|Ax-b\|_2=\min\right\}$$

and assume that

$$\|Cx_{A,C}-d\|_2>\gamma, \quad \operatorname{rank}\begin{bmatrix}A\\C\end{bmatrix}=n, \tag{1.2}$$

then there is a unique solution to (1.1). Under the assumption (1.2) the unique solution x to (1.1) satisfies the generalized normal equations

$$(A^\top A+\lambda C^\top C)x=A^\top b+\lambda C^\top d, \tag{1.3}$$

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where the parameter λ is determined by the secular equation

$$\|Cx-d\|_2=\gamma.$$

In the following we will denote $Q(A,C)=(A^\top A+\lambda C^\top C)^{-1}$. Then

$$x=Q(A,C)\left(A^\top b+\lambda C^\top d\right).$$

A particularly simple but important case of (1.1) is when

$$C=I_n \quad \text{and} \quad d=\mathbf{0}, \tag{1.4}$$

where I_n is the $n\times n$ identity matrix. The corresponding *standard form* of GLSQI (LSQI) is given by

$$\min_{x\in\mathbb{R}^n}\|Ax-b\|_2 \quad \text{subject to } \|x\|_2\le\gamma, \tag{1.5}$$

where $\gamma>0$, $A\in\mathbb{R}^{m\times n}$ and $b\in\mathbb{R}^m$. If $\|A^\dagger b\|\le\gamma$, where $A^\dagger$ is Moore–Penrose inverse [3,1,4] of A, then $A^\dagger b$ is the unique solution of (1.5), i.e., LSQI reduces to the unconstrained linear least squares problem. Therefore, throughout this paper we assume that $\|A^\dagger b\|\ge\gamma$, where γ is a positive number in (1.5). Under the above condition, the unique solution of (1.5) satisfies the following equation

$$(A^\top A+\lambda I_n)x=A^\top b,$$

where λ is a positive parameter such that the constraint $\|x\|_2=\gamma$ holds. In the following we denote $P(A,\lambda)=(A^\top A+\lambda I_n)^{-1}$. Thus $x=P(A,\lambda)A^\top b$.

In numerical analysis, condition number is an important research topic, which measures the *worst-case* sensitivity of an input data to *small* perturbations. A problem with large condition number is called *ill-posed* problem [5]. Also Demmel pointed that the distance of a problem to ill-posed sets is the reciprocal of its condition number. For the comprehensive review on condition numbers, we refer to the recent book [6].

Let V and W be two Banach spaces and U an open subset of V. Considering an operator $f:U\to W$, if, for an $x\in U$, there exists a bounded linear operator $\mathcal{A}_x:V\to W$ such that

$$\lim_{h\to 0}\frac{\|f(x+h)-f(x)-\mathcal{A}_x(h)\|_W}{\|h\|_V}=0,$$

then f is said to be Fréchet differentiable at x and $\mathcal{A}_x$ is called the Fréchet derivative of f at x.

To the best of our knowledge a general theory of condition numbers was first given by Rice in [7]. Let $\phi:\mathbb{R}^s\to\mathbb{R}^t$ be a mapping, where $\mathbb{R}^s$ and $\mathbb{R}^t$ are the usual s- and t-dimensional Euclidean spaces equipped with some norms, respectively. If ϕ is continuous and Fréchet differentiable in the neighborhood of $a_0\in\mathbb{R}^s$ then, according to [7], the *relative normwise condition number* of a_0 is given by

$$\operatorname{cond}^{\phi}(a_0):=\lim_{\varepsilon\to 0}\sup_{\|\Delta a\|\le\varepsilon}\left(\frac{\|\phi(a_0+\Delta a)-\phi(a_0)\|}{\|\phi(a_0)\|}\Big/\frac{\|\Delta a\|}{\|a_0\|}\right)=\frac{\|\mathrm{d}\phi(a_0)\|\,\|a_0\|}{\|\phi(a_0)\|}, \tag{1.6}$$

where $\mathrm{d}\phi(a_0)$ is the Fréchet derivative of ϕ at a_0. Condition number can tell us the loss of the precision in finite precision computation of a problem. With the backward error of a problem, we have the following rule of thumb [8]

$$\text{forward error} \lesssim \text{condition number} \times \text{backward error},$$

which can bound the relative error of the computed solution.

When the data is sparse or badly scaled, normwise condition numbers defined by (1.6) may overestimate the true error since measuring perturbations by norms allows large perturbations on small entries of the data, further more zeros entries may be perturbed to non zeros. From 1980s, componentwise perturbation analysis has been proposed, see papers[9–12] and an early survey [13] and references therein. In fact, most error bounds in LAPACK [14] are based on componentwise perturbation analysis. There are two kinds of condition numbers in componentwise analysis: the mixed condition numbers and componentwise condition numbers [10]. The mixed condition numbers use the componentwise error analysis for the input data, while the normwise error analysis for the output data. On the other hand, the componentwise condition numbers use the componentwise error analysis for both input and output data. Mixed and componentwise condition numbers had been introduced and studied for linear least squares problem (LS) [9], structured LS problem [15], LS problem involving Kronecker products [16], Tikhonov regularization [17] and structured Tikhonov regularization problem [18].

Malyshev in [19] took a unified theory to study the normwise condition numbers for Tikhonov regularization, GLSQI, LSQI and linear least squares with equality constraints when only perturbations on the coefficient matrices are considered. The normwise condition numbers under the circumstance that there are perturbations both on the matrices and right hand vectors are not investigated up to now.

In this paper, we will study the conditioning of the following linear mappings Φ and ψ. We introduce the notation $\mathrm{vec}(A)$ firstly. For any matrix $A=[a_1\ a_2\ \cdots\ a_n]\in\mathbb{R}^{m\times n}$, we define $\mathrm{vec}(A)\in\mathbb{R}^{mn}$ by $\mathrm{vec}(A)=[a_1^\top\ a_2^\top\ \cdots\ a_n^\top]^\top$, by stacking the columns of A. So for GLSQI, we introduce the mapping as follows

$$\Phi:\mathbb{R}^{mn}\times\mathbb{R}^{pn}\times\mathbb{R}^{m}\times\mathbb{R}^{p}\to\mathbb{R}^{n}$$