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Existence and upper semicontinuity of random attractors for non-autonomous stochastic strongly damped sine–Gordon equation on unbounded domains ☆

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ABSTRACT

In this paper we study the asymptotic behavior of solutions of the non-autonomous stochastic strongly damped sine–Gordon equation driven by multiplicative noise defined on an unbounded domain. First we introduce a continuous cocycle for the equation and establish the pullback asymptotic compactness of solutions. Second we consider the existence and upper semicontinuity of random attractors for the equation.

1. Introduction

A random attractor, introduced by Crauel et al. [1] and Crauel and Flandoli [2] to capture the essential dynamics with possibly extremely wide fluctuations, is a generalization of the global attractor of autonomous differential equations for random dynamical systems. In recent years, random attractors for non-autonomous stochastic partial differential equations have been investigated in [3,4] in bounded domains and in [5–9] on unbounded domains.

In this paper, we consider the existence of a random attractor for the following non-autonomous stochastic strongly damped sine–Gordon equation with multiplicative noise defined in the entire space $\mathbb{R}^n$ ($n \in \mathbb{N}$):

$$u_{tt} - \alpha \Delta u_t - \Delta u + u_t + \lambda u + \mu \sin u = g(x,t) + cu \circ \frac{dW}{dt}, \tag{1.1}$$

with the initial value conditions

$$u(\tau, x) = u_\tau(x), \qquad u_t(\tau, x) = u_{1\tau}(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where Δ is the Laplacian with respect to the variable $x \in \mathbb{R}^n$; $u = u(t,x)$ is a real function of $x \in \mathbb{R}^n$ and $t \geqslant \tau$, $\tau \in \mathbb{R}$; $\alpha > 0$ is the strong damping coefficient; $\lambda > 0$, μ and c are constants; $g(x,\cdot)$ is a given function in $L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$; $\circ$ denotes the Stratonovich sense in the stochastic term; $W(t)$ is a two-sided real-valued Wiener process on a probability space. If g does not depend on time, then we call Eq. (1.1) an autonomous equation.

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Sine–Gordon equation describes the dynamics of continuous Josephson junctions (see [10]) and has a wide range of applications in physics. Recently, several authors [11–13] studied the attractors for a stochastic damped sine–Gordon equation on a bounded domain, but not on unbounded domains. So far as we know, there were no results on random attractors for non-autonomous stochastic strongly damped sine–Gordon equation (1.1) on unbounded domains. In general, the existence of global random attractor depends on some kind compactness (see, e.g., [14,1,2]). The main difficulty of this paper is to prove the asymptotic compactness of solutions, because Sobolev compact embedding is lost for unbounded domain. In order to overcome the difficulty, we use uniform estimates on the tails of solutions outside a bounded ball in $\mathbb{R}^n$, then decompose the solutions in a bounded domain in terms of eigenfunctions of negative Laplacian as in [15,16]. On the other hand, the cases of $g(x,t)$ depending on time and multiplicative noise $cu \circ \frac{dW}{dt}$ are of great physical interest. It is therefore important to investigate the existence of attractors for Eq. (1.1). In these cases, we need two separate parametric spaces to deal with the deterministic perturbations $g(x,t)$ as well as the stochastic perturbations $cu \circ \frac{dW}{dt}$.

This paper is organized as follows. In the next section, we recall some basic settings for Eq. (1.1) and define a continuous cocycle in $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. In Section 3, we first derive all necessary uniform estimates of solutions, then prove the existence and uniqueness of tempered pullback random attractors for Eq. (1.1). In Section 4, we prove the upper semicontinuity of random attractors as the coefficient of the white noise term tends to zero.

Throughout this paper, we use $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ to denote the norm and the inner product of $L^2(\mathbb{R}^n)$, respectively. The norm of a Banach space X is generally written as $\|\cdot\|_X$. The letters $\mathbb{c}$ and $\mathbb{c}_i$ $(i = 1, 2, \ldots)$ are used to denote positive constants whose values are not significant in the context.

2. Mathematical settings

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the standard probability space where $\Omega = \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω, and $\mathcal{P}$ is the Wiener measure on $(\Omega,\mathcal{F})$ (see [17]). There is a classical group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\Omega,\mathcal{F},\mathcal{P})$ which is defined by $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$, for $\omega \in \Omega$, $t \in \mathbb{R}$. Then $(\Omega,\mathcal{F},\mathcal{P},\{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system.

Denote the inner and norm of $L^2(U)$ as $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively, and $E(U) = H^1(U)\times L^2(U)$, $U \subseteq \mathbb{R}^n$, endowed with the usual norm

$$\|Y\|_{H^1(U)\times L^2(U)} = \left(\|\nabla u\|^2 + \|u\|^2 + \|v\|^2\right)^{\frac{1}{2}} \quad \text{for } Y = (u,v)^\top \in E(U), \tag{2.1}$$

where $\top$ stands for the transposition. We define a new norm $\|\cdot\|_{E(U)}$ by

$$\|Y\|_{E(U)} = \left(\|v\|^2 + \left(\delta^2 + \lambda - \delta\right)\|u\|^2 + \left(1 - \alpha\delta\right)\|\nabla u\|^2\right)^{\frac{1}{2}}, \tag{2.2}$$

where δ satisfied (3.6), for $Y = (u,v)^\top \in E(U)$. It is easy to check that $\|\cdot\|_{E(U)}$ is equivalent to the usual norm $\|\cdot\|_{H^1(U)\times L^2(U)}$ in (2.1).

For our purpose, it is convenient to convert the problem (1.1)–(1.2) into a deterministic system with a random parameter, and then show that it has a cocycle on $E(\mathbb{R}^n)$ over $\mathbb{R}$ and $(\Omega,\mathcal{F},\mathcal{P},\{\theta_t\}_{t\in\mathbb{R}})$. We identify $\omega(t)$ with $W(t)$, i.e., $\omega(t) = W(t)$, $t \in \mathbb{R}$. Consider Ornstein–Uhlenbeck equation $dz + \alpha z dt = dW(t)$, and Ornstein–Uhlenbeck process $z(\theta_t\omega) = -\alpha\int_{-\infty}^0 e^{\alpha s}(\theta_t\omega)(s)ds$, $t \in \mathbb{R}$. From [18,19], it is known that the random variable $|z(\theta_t\omega)|$ is tempered, and there is a θ_t-invariant set $\widetilde{\Omega} \subset \Omega$ of full $\mathcal{P}$ measure such that $z(\theta_t\omega)$ is continuous in t for every $\omega \in \widetilde{\Omega}$.

Lemma 2.1 (*See [12]*). *For the Ornstein–Uhlenbeck process $z(\theta_t\omega)$, we have the following results*

$$\lim_{t\to\infty}\frac{|z(\theta_t\omega)|}{|t|} = 0, \tag{2.3}$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t z(\theta_s\omega)ds = 0, \tag{2.4}$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t |z(\theta_s\omega)|ds = \frac{1}{\sqrt{\pi\alpha}}, \tag{2.5}$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t |z(\theta_s\omega)|^2 ds = \frac{1}{2\alpha}. \tag{2.6}$$

Let $v(t,\tau,\omega) = u_t(t,\tau,\omega) + \delta u(t,\tau,\omega) - cu(t,\tau,\omega)z(\theta_t\omega)$, where δ is as in (2.2), then (1.1)–(1.2) can be rewritten as the equivalent system with random coefficients but without white noise