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A rational high-order compact difference method for the steady-state stream function–vorticity formulation of the Navier–Stokes equations

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ABSTRACT

A rational high-order compact (RHOC) finite difference (FD) method on the nine-point stencil is proposed for solving the steady-state two-dimensional Navier–Stokes equations in the stream function–vorticity form. The resulting system of algebra equations can be solved by using the point-successive over- or under-relaxation (SOR) iteration. Numerical experiments, involving two linear and two nonlinear problems with their analytical solutions and two flow problems including the lid driven cavity and backward-facing step flows, are carried out to validate the performance of the newly proposed method. Numerical solutions of the driven cavity problem with different grid mesh sizes (maximum being 513×513) for Reynolds numbers ranging from 0 to 17500 are obtained and compared with some of the accurate results available in the literature.

1. Introduction

The Navier–Stokes equations are highly nonlinear and are very difficult to solve, especially when the approximate solutions are required to have a high accuracy. Therefore, accurate and efficient difference representations of the Navier–Stokes equations are of vital importance.

The main purpose of this paper is to present a new fourth-order compact finite difference (FD) scheme to solve the steady two-dimensional (2D) incompressible Navier–Stokes equations in the stream function–vorticity formulation. High order finite difference methods, which are classified as *wide* or *compact* type, have been used since they have some advantages when compared with the traditional second-order central differences (CD) methods or the first-order upwind differences (UD) methods [1–4]. The high order FD methods of wide type are obtained discretizating the equations by fourth order central differences which results in a wider computational stencil. The high-order compact type (HOC) FD methods are constructed with the aims of having a narrower stencil and maintaining the high order accuracy [5–22]. These methods, which are computationally efficient and yield highly accurate numerical solutions, can be obtained for the stream function–vorticity formulations of the Navier–Stokes equations [2,15,16,18,19,21–23,3]. Gupta et al. [18] applied their compact fourth-order nine-point compact FD formula developed for partial differential equations of elliptic type [14] to the solution of the steady 2D Navier–Stokes equations in the stream function–vorticity form and solved the driven cavity

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problem for Reynolds numbers up to $Re = 2000$ with a maximum of 41×41 grid mesh using point-SOR type of iteration. This study was followed more recently by similar research by Spotz and Carey [15], and Li et al. [16]. Spotz and Carey [15] developed a compact fourth order method on a nine-point computational stencil for the steady 2D stream function–vorticity formulation of the Navier–Stokes equations. They solved the square driven cavity flow up to $Re = 1000$ with a maximum of 41×41 grid mesh using the generalized minimal residual method and successive approximations. Li et al. [16] presented a compact fourth order scheme for the stream function–vorticity formulations, which is strictly within the 3×3 computational stencil and had a faster convergence than that of Gupta [18]. They solved the cavity flow with a grid size of 129×129 for $Re \leq 7500$ using point-SOR iterations and found that SOR pointwise iteration does not work when $Re \geq 9000$. In [23], Zhang employed fourth-order compact finite difference schemes [14] with multigrid techniques to simulate the two-dimensional square driven cavity flow for Reynolds numbers up to $Re = 7500$ with a maximum of 129×129 grid mesh. Dennis and Hudson [21] developed the same scheme as in Ref. [14] using another approach. This method is a two-dimensional version of the methods of exponential type and uses the Numerov approximation. They solved the problem of natural convection in a square cavity with two vertical sidewalls maintained at different temperatures and obtained results up to $Ra = 10^5$ (Ra is the non-dimensional Rayleigh number). Kalita et al. [24] computed the laminar solution of the problem with a fourth-order accurate HOC scheme originally proposed by Spotz and Carey [15] for the stream function–vorticity form of the 2D steady Navier–Stokes equations. They used conjugate gradient and hybrid biconjugate gradient stabilized algorithms and solved the natural convection in a square cavity up to $Ra = 10^7$. Recently, Erturk and Gökçöl [3] developed a new fourth-order compact formulation, whose difference with Refs. [15,16,18,21,22] is that the final form of the HOC formulation is in the same form of the Navier–Stokes equations such that any iterative numerical method used for Navier–Stokes equations, can be easily applied to this new HOC formulation, since the final form of the presented HOC formulation is in the same form with the Navier–Stokes equations. They used a fine grid mesh of 601×601, as it was suggested by Erturk et al. [25] in order to be able to compute and obtain a steady solution for high Reynolds numbers and solved the driven cavity flow for Reynolds number up to $Re = 20\,000$.

In this work, we introduce a new fourth-order compact (4OC) method. The main difference of the new 4OC method with the HOC methods proposed in [15,18,22] is in the way that the fourth-order compact scheme is obtained for the 2D model problems (i.e. convection–diffusion equations). The key property with the present 4OC scheme is that it allows the point-successive over- or under-relaxation (SOR) iteration for low-to-high Reynolds numbers. For the 2D model problems with constant convection coefficient, in order to determine the computational stencil coefficients appropriate for the desired order of accuracy, the new 4OC method integrates exactly any linear combination of the functions $\{1, x, y, xy, x^2, y^2, x^2y, xy^2, x^3, y^3, x^2y^2, x^4, y^4\}$. For the variable convection coefficient problem, the 4OC scheme is achieved based on the 4OC scheme proposed for the constant convection coefficient problems and a practical technique, named the reminder term modification approach [19]. As the basis of a discretization method for the incompressible, 2D, steady-state flow problems, the 4OC formulation newly proposed for the 2D convection–diffusion equation is applied to the stream function–vorticity formulation of the Navier–Stokes equations. To demonstrate the performance of this new formulation, two linear and two non-linear problems with their analytical solutions are computed at different grid mesh sizes and the lid driven cavity flow problem for Reynolds numbers up to $Re = 17\,500$ with a maximum of 513×513 grid mesh are solved and compared with some of the accurate results accessible in the literature.

In the next section, we first present a rational fourth-order compact (R4OC) scheme on a 3×3 stencil for the 2D convection–diffusion equation with constant convection coefficient; then we develop a nine-point R4OC scheme of the 2D problem with variable convection coefficient. Applications of the currently proposed 4OC formulation to the Navier–Stokes equations in the stream function–vorticity formulation follow in Section 3. Convergence studies to validate fourth-order accuracy of the proposed R4OC method for four problems with their analytical solutions are given and the numerical solutions of the lid driven cavity and backward-facing step flow problems are performed in Section 4. Finally, Section 5 is devoted to some concluding remarks.

2. Fourth-order compact FD method for model problem

Consider the two-dimensional non-homogeneous convective diffusion model problem

$$-au_{xx} - bu_{yy} + p(x, y)u_x + q(x, y)u_y = f(x, y) \tag{1}$$

where a and b are constants and p and q vary spatially, and f is a sufficiently smooth function with respect to x and y. This equation is also the principal equation for the 2D steady Navier–Stokes equations with constant viscosity in both the primitive variables formulation and the stream function–vorticity formulation. Let the step-length in the x-direction be $h = 1/n$ and in the y-direction be $k = 1/m$, where n and m are the numbers of subdivisions in the x- and y-directions respectively.

There are several strategies [22,2,14,15,19] to derive 4OC schemes for Eq. (1). Gupta et al. [14] derived a 4OC FD scheme based on the truncated Taylor series expansions. Their procedures give the approximate value of a function at a mesh point as a linear combination of the analytic solutions of the partial differential equation. The FD formula is obtained by collocation over a set of mesh points surrounding the given mesh point for which the difference formula is derived. The procedure to combine terms and to simplify formula is straightforward but extremely tedious, especially for derivation in higher dimensions. Another technique to obtain HOC formulae consists of considering a particular equation and

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