Computers and Mathematics with Applications 73 (2017) 1505–1523

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# Extended framework of Hamilton's principle for thermoelastic continua

Jinkyu Kim [*]

*School of Architecture and Architectural Engineering, Hanyang University, 55 Hanyangdaehak-ro, Sangnok-gu, 15588, Kyeonggi-do, Republic of Korea*

ARTICLE INFO

*Article history:*
Received 22 August 2016
Received in revised form 7 January 2017
Accepted 28 January 2017
Available online 13 February 2017

*Keywords:*
Extended framework of Hamilton's principle
Variational formulation
Thermoelasticity
Space–time finite element method

ABSTRACT

Based upon the extended framework of Hamilton's principle, a variational formulation for fully coupled thermoelasticity is presented. The resulting formulation can properly account for all the governing differential equations as well as initial/boundary conditions. Thus, it provides the basis for a class of unified space–time finite element methods. By employing bar elements in one-dimensional space along with linear shape functions temporally, the simplest space–time finite element method is presented herein with representative examples for its validity.



## 1. Introduction

In current practice, most time-dependent phenomena are numerically addressed with discrimination on space and time: standard finite element methods are used in space to reduce a set of differential equations in time, then, time-integration methods are employed for numerical solutions. Such space–time disparity may originate from the absence of a variational framework that can properly account for initial conditions.

For discrete particle dynamics, Hamilton [1,2] presented variational formulation with the concept of stationary action, where the functional action is defined as the integration over time of the Lagrangian function of the system. This Hamilton's principle is, of course, one of pillars of classical dynamics and has broad applicability in mathematical physics and engineering [3–7]. However, there are two main difficulties. First, it cannot incorporate initial conditions properly. More specifically, it requires that the variations at the beginning and end of the time interval vanish, which exploits time-boundary conditions rather than initial conditions. These temporal boundary conditions in Hamilton's principle represent that functions are known at the beginning and the end of the time interval. However, one cannot assume that the position of each particle at the end of the time interval is known, and in general, this is the primary objective in initial value problems. This critical and philosophical weakness in Hamilton's principle is called end-points constraint issue, and a thorough historical review about this weakness can be found in the chapter 5 of [8]. The second one is the restriction to conservative systems. With Rayleigh's dissipation [9], Hamilton's principle can account for dissipative irreversible dynamical systems. While this approach can lead to proper governing differential equations in weak formalism, it is not entirely satisfactory as a variational statement. In particular, the variation of Rayleigh's dissipation function enters in *ad hoc* manner. Also, the end-points constraint issue remains, as in the original Hamilton's principle for the conservative system.

* Fax: +82 31 436 8169.
*E-mail address:* jinkyu29547@gmail.com.

http://dx.doi.org/10.1016/j.camwa.2017.01.021

Recently, in order to resolve critical weakness of end-points constraint in the Hamilton's principle, the extended framework of Hamilton's principle (EHP) was developed for elastic/viscoplastic continuum dynamics [10], single-degree-of-freedom/multi-degrees-of-freedom linear elastic systems [11–13], and pure heat diffusion [14]. This new variational framework utilizes a mixed Lagrangian formulation (MLF, [15–22]), and the external specification of initial conditions added in the variation of action. It cannot reside in a complete variational principle, because Rayleigh's dissipation is still required for irreversible dissipative systems, and it cannot strictly follow a variational statement. However, the framework is quite simple and it initiates the implementation of unified space–time finite element approach with proper use of the initial conditions. Based upon the previous success in elastodynamics [10] and pure heat diffusion [14] with this variational approach, as a prototype extension, fully coupled dynamic irreversible thermoelasticity of continua is addressed in the present paper, where the Maxwell–Chester version of the extended Fourier's law [23,24] are taken into account for second sound effects among others [25–28]. The paper firstly presents a variational formulation of thermoelasticity, where all the governing differential equations are recovered from the corresponding Euler–Lagrange equations along with proper use of initial and boundary conditions. Then, armed with this theoretical basis, the simplest form of unified space–time finite element method is developed to demonstrate key ideas of consistent discretization scheme over both spatial and temporal domains. Finally, several examples in semi-infinite domain are provided to validate this unified space–time finite element approach with comparison to the known analytical solutions [29,30].

## 2. Variational formulation for thermoelasticity

In this section, a new variational formulation of thermoelasticity stemming from the extended framework of Hamilton's principle (EHP) is presented. It begins with defining appropriate primary mixed variables and the corresponding Lagrangian, potential function, and Rayleigh's dissipation. In particular, mixed variables are defined in impulsive forms as

$$\theta(t) = \int_0^t T(s)\, ds; \qquad \dot{\theta} = T \tag{1}$$

$$H_i(t) = \int_0^t q_i(s)\, ds; \qquad \dot{H}_i = q_i \tag{2}$$

$$J_{ij}(t) = \int_0^t \sigma_{ij}(s)\, ds; \qquad \dot{J}_{ij} = \sigma_{ij} \tag{3}$$

where $\theta$ is the impulse of temperature $T$, while $H_i$ and $J_{ij}$ represent the impulse of heat flux $q_i$ and the impulse of elastic stress $\sigma_{ij}$, respectively. Thus, in this formulation, the purely elastic stress $\sigma_{ij}$ is written as $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$ in terms of the constitutive tensor $C_{ijkl}$ and the total strain $\varepsilon_{kl}$.

Over the domain $\Omega$ of a medium enclosed by the surface $\Gamma$, fully coupled thermoelasticity problems can be captured by following three functions in the EHP

$$l = \frac{1}{2}\rho\,\dot{u}_i\,\dot{u}_i + \frac{1}{2}A_{ijkl}\dot{J}_{ij}\dot{J}_{kl} - \left(\dot{J}_{ij} - \beta_{ij}\,\dot{\theta}\right)\varepsilon_{ij} + \frac{\rho\,c}{2\,T_0}\dot{\theta}^2 + \frac{\tau_0}{2\,T_0}d_{ij}\,\dot{H}_i\,\dot{H}_j - \frac{1}{T_0}\dot{\theta}_{,i}\,H_i \tag{4}$$

$$V = \int_\Omega \left(\frac{1}{T_0}\bar{\psi}\,\theta + \bar{f}_i\,u_i\right) d\Omega + \int_{\Gamma_\tau} \bar{\tau}_i\,u_i\,d\Gamma - \int_{\Gamma_q} \frac{1}{T_0}\bar{q}\,\theta\,d\Gamma \tag{5}$$

$$\varphi = \frac{1}{2}\frac{1}{T_0}d_{ij}\,\dot{H}_i\,\dot{H}_j \tag{6}$$

where $l$, $V$, and $\varphi$ represent Lagrangian density, potential function, and Rayleigh's dissipation, respectively.

In Eqs. (1)–(6), the comma notation represents a partial derivative with respect to one of the spatial coordinates, while superposed dot represents a time derivative. Also, $T_0$ denotes the initial absolute temperature at the free-stress state, while $T$ then becomes the temperature change from that state. For other notations, $\varepsilon_{ij}$ represents a total strain, $A_{ijkl}$ is the inverse of the usual constitutive tensor $C_{ijkl}$ and $d_{ij}$ is the inverse of the conductivity $k_{ij}$, while $\beta_{ij}$ is a thermal moduli tensor. Additionally, $\rho$ is the mass density, $c$ is the specific heat coefficient at constant strain, $\tau_0$ is a relaxation time for the Maxwell–Chester heat conduction law, $\bar{f}_i$ is a specified body force density, and $\bar{\tau}_i$ is a traction specified on the portion of the surface $\Gamma_\tau$. Furthermore, $\bar{q}$ is a specified normal heat flux on the portion of surface $\Gamma_q$, and $\bar{\psi}$ is a specified body heat source rate per unit volume.

With these functions specified in Eqs. (4)–(6), in the EHP, the first variation of the action integral is newly defined as

$$\begin{aligned}\delta I_{NEW} = &-\int_0^t\int_\Omega \delta l\,d\Omega\,ds - \int_0^t \delta V\,ds + \int_0^t\int_\Omega \frac{\partial\varphi}{\partial\dot{H}_j}\,\delta H_i\,d\Omega\,ds + \int_\Omega \left[\rho\,\hat{\dot{u}}_i\,\delta\hat{u}_i\right]_0^t d\Omega \\ &+ \int_\Omega \left[\frac{1}{T_0}\hat{\Psi}\,\delta\hat{\theta}\right]_0^t d\Omega - \int_{\Gamma_q}\left[\frac{1}{T_0}\hat{Q}\,\delta\hat{\theta}\right]_0^t d\Gamma - \int_\Omega \left[\frac{1}{T_0}\left(d_{ij}\,\hat{H}_j + \hat{\theta}_{,i}\right)\delta\hat{H}_i\right]_0^t d\Omega\end{aligned} \tag{7}$$