Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Construction of equilibrated singular basis functions without a priori knowledge of analytical singularity order

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ARTICLE INFO

Article history:
Received 31 August 2016
Received in revised form 25 November 2016
Accepted 5 February 2017
Available online xxxx

Keywords:
Singularity
Harmonic
Bi-harmonic
Equilibrated basis functions

ABSTRACT

In this paper a novel method is presented to construct singular basis functions for solving harmonic and bi-harmonic problems with weak singularities. Such bases are found without any knowledge of the singularity order. The singular bases are constructed by choosing a series as a tensor product of Chebyshev polynomials and trigonometrical functions, in radial and angular directions respectively, and applying weak form of the governing equation. With such features, the singular bases are categorized as the equilibrated basis functions. The constructed singular functions can be utilized as a complementary part to the smooth part of the approximation in the solution of problems with singularities. To demonstrate the efficiency of employing such singular bases, they are used in a boundary node method. Through the solution of some examples, selected from the well-known literature, the capability of the method is shown. It will be demonstrated that the main function and its derivatives are excellently approximated at very close neighborhood of the singular point. The method may especially be found useful for those who research on the eXtended Finite Element Method or similar ideas.

1. Introduction

Solution of problems with singularities has always been a subject of interest as the singularities are usually considered as the critical points of weakness or as the locations for stress/flux concentration in a continuum. The concept of singularity is related to a discontinuity in the solution function or its derivatives. The latter is mostly known as the weak singularity which makes a sort of flux concentration (e.g. stress concentration). From a mathematical point of view, the derivatives of the solution function tend to infinity in the adjacency of such points and thus the ordinary smooth approximations, usually implemented in both the mesh-based and mesh-less methods, cannot adapt to such conditions. The most widely used technique implemented to localize the pollution effect of the singularities is the adaptive mesh generation or refinement [1]. This however poses an extra expense to the solution procedure. Another popular technique is to add the so called "singularity inducing terms" into the set of bases or shape functions. This has been the main concept in developing the eXtended Finite Element Method (XFEM) [2] and some mesh-less methods [3,4]. The latter idea is of much lower cost compared to the first one, but it needs a priori knowledge of the singularity order. This necessitates analytically solving a partial differential equation (PDE), usually in a polar coordination system on an infinite domain. Such analytical solutions may not be easily found for many cases and this has been the main disadvantage of the aforementioned approach even in recent works [5].

An alternative idea to the aforementioned analytical solution for finding the singular terms is obviously a numerical approach. One of the most serious obstacles in this way is that the singularity inducing terms cannot be expressed by the

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http://dx.doi.org/10.1016/j.camwa.2017.02.004

smooth basis or shape functions usually implemented in the numerical methods. This necessitates the use of approximate singular functions. The decision on the extent of the domain and the approximation used for satisfying the PDE is another hurdle in this way.

Apart from the necessity of using singular functions, the required process has much in common with the recently developed numerical technique in [6,7] in which a set of so called Equilibrated Basis Functions (EqBFs), using smooth polynomials, is found on a fictitious domain. The use of EqBFs in a numerical solution, while focusing on non-homogeneous media, is considered as a continuation of the studies on the Exponential Basis Functions (EBFs) [8–16] whose main focus was on PDEs with constant coefficients. Therefore, the EqBFs may be viewed as the generalized EBFs suitable for PDEs with non-constant coefficients. The bases in the EqBFs are constructed by the summation of a set of primary bases and through a weighted residual approach applied to the governing PDE. The PDE is thus approximately satisfied which leads to facilitating the extraction of the EqBFs for various cases. This advantage, along with the completeness of the primary bases and also the possibility of using predefined one-dimensional integrals (see [6,7]), makes the method efficient for being used in many PDEs. However, due to the use of smooth polynomials in the construction of the EqBFs, proposed in [6,7], problems with singularities may not easily be dealt with. Nevertheless, the capability of the method in dealing with PDEs defined in polar coordinates, as special cases of PDEs with non-constant coefficients, paves the way for studies on problems with singularities.

The main objective in this paper is to present a technique, similar to its counterpart for finding the EqBFs, to equip the numerical method with some additional singular bases to accurately model problems with singularities. As will be seen later in this paper, although the overall feature of the method follows that of the former version, i.e. using smooth primary bases, a simple mapping technique will help us to convert the bases to some singular ones. The primary singular bases are then used in a weighted residual approach to find the final equilibrated singular bases. These bases are used in conjunction with those previously found for smooth problems. For the sake of consistency and compatibility with our previous studies in [6,7], the former EqBFs are used to reconstruct the smooth part of the solution, assuming that the solution may be split into smooth and singular parts (the reader may refer to [12,17] for a rather similar experience in using local/global forms of EBFs mixed with some additional singular functions). Other methods capable of reconstruction of the smooth part of the solution, e.g. the finite element method (FEM), may also be used. In that case the combined methods may have much in common with the idea of XFEM [2]. However, such a study is beyond the scope of this paper.

The main novelty of the presented technique lies in the fact that the additional bases are chosen without a priori knowledge of the singularity order (which was the case in the similar methods mentioned earlier). It will be shown that they can automatically identify and reproduce the needed singular terms.

This paper is organized as follows. In Section 2 a brief review of the EqBFs will be given. Then in Section 3 the method is reformulated in light of the new concept while the new bases are constructed for harmonic and bi-harmonic PDEs. In Section 4 some well-known numerical examples are solved to demonstrate the capabilities of the new technique. The paper will be closed in Section 5 by some concluding remarks.

2. Equilibrated basis functions: an overview

In order to give a brief overview of the concept of using equilibrated bases functions (EqBFs), we begin with assuming a generic partial differential equation (PDE), now with smooth solution on a simply connected domain (see also [6,7]). The PDE, here in 2D space and its corresponding boundary conditions are considered as below,

$$\begin{aligned} &Lu = 0, \quad \text{in } \Omega, \\ &L_B u = u_B, \quad \text{on } \Gamma = \partial\Omega. \end{aligned} \tag{1}$$

Here L is the PDE operator, u is the potential field and L_B applies the boundary conditions u_B along the domain boundaries. The solution of the PDE is approximated by a linear combination of some primary basis functions (f_n) which do not necessarily satisfy the PDE individually,

$$u \simeq \hat{u} = \sum_{n=1}^{N} f_n c_n = \mathbf{f}^T \mathbf{c}. \tag{2}$$

A relation between the values of the unknown coefficients c_n should be found such that the mentioned series satisfies the homogeneous PDE. This is performed through the application of a weighted residual integration of the PDE over the whole solution domain. Note that by writing the integrals over the whole computational domain, the approximated field $\hat{u}$ will be qualified for being used in a boundary node/integral method since the remaining task for reaching an approximation solution for the problem defined in (1) is the satisfaction of the boundary conditions. To this end, and for convenience, the simply connected domain Ω is embedded inside a fictitious domain Ω_0 of rectangular shape, circumscribing Ω, as in Fig. 1.

The weighted integration is thus written as,

$$\left(\int_{\Omega_0} \mathbf{w} L \mathbf{f}^T \, d\Omega \right) \mathbf{c} = \mathbf{A}\mathbf{c} = \mathbf{0}. \tag{3}$$

In the above equation $\mathbf{w}$ is the vector of the chosen weight functions. The logic underling (3) is that when the PDE is considered to govern over Ω_0, it governs over any subdomain of Ω_0 including Ω. This, however, needs defining some