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# Conjugate gradient least squares algorithm for solving the generalized coupled Sylvester matrix equations[☆]

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ABSTRACT

This paper discusses the conjugate gradient least squares algorithm for solving the generalized coupled Sylvester matrix equations $\sum_{j=1}^{q} \boldsymbol{A}_{ij}\boldsymbol{X}_j\boldsymbol{B}_{ij} = \boldsymbol{F}_i$, $i = 1, 2, \ldots, p$. We prove that if this system is consistent then the iterative solution converges to the exact solution and if this system is inconsistent then the iterative solution converges to the least squares solution within the finite iteration steps in the absence of the roundoff errors. Also by setting the initial iterative value properly we prove that the iterative solution converges to the least squares and minimum-norm solution.



## 1. Introduction

Some different matrix equations are encountered in the system theory [1–3], control theory [4,5] and stability analysis [6]. How to obtain the solutions of these matrix equations become an important topic in applied mathematics and engineering area [7–10]. Some different iterative techniques for solving matrix equations were studied.

Recently, by combining the classic iteration method and the hierarchical identification principle, Ding and his coworkers offer a class of iterative algorithms to solve the matrix equations [11–14]. The gradient-based and least squares based iterative methods are the spacial cases of this class of iterative methods. For examples, by using the hierarchical identification principle and the gradient-based iterative method of the simple matrix equations, the gradient-based iterative algorithms were established for solving some linear matrix equations [15,16] and some coupled matrix equations [17,18]. This method was extended to solving some more complicated matrix equations [19–22].

Although some complicated matrix equations can be solved through decomposing them into some small matrix equations by using the hierarchical identification principle, the range of the convergence factor was unsettled. To settle this question and by using the gradient search principle and minimizing the weighted object function, Zhou developed the weighted least squares iterative algorithms for solving the general coupled matrix equations [23,24]. This method can determine the range and the optimum choice of the convergence factor. The gradient-based and least squares based iterative methods are the special cases of this iterative method [4,25,26].

Another active method to solve the matrix equations is the conjugate gradient method [27–29]. By introducing an inner product and generating an orthogonal basis of the residual matrix, this method can obtain the solution within the

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finite iteration steps in the absence of the roundoff errors. Through decades, this method has become more and more popular [30–33]. For example, the skew-symmetric solution of matrix equation $\boldsymbol{AXB} = \boldsymbol{C}$ was established [34]. The conjugate gradient squared and bi-conjugate gradient stabilized finite iterative methods were suggested for the Sylvester-transpose and periodic Sylvester matrix equations [35]. To obtain the reflexive and anti-reflexive solutions, with proper operations, two conjugate gradient algorithms for solving the matrix equation $\boldsymbol{A}_1\boldsymbol{X}_1\boldsymbol{B}_1 + \boldsymbol{A}_2\boldsymbol{X}_2\boldsymbol{B}_2 = \boldsymbol{C}$ was established [36]. To improve the performance of the suggested algorithm, a preconditioned nested splitting conjugate gradient iterative method was presented for solving the generalized Sylvester equation [37] and an alternating direction method for matrix equation $\boldsymbol{AXB} + \boldsymbol{CYD} = \boldsymbol{E}$ was suggested in [38].

The conjugate gradient algorithms for solving the coupled matrix equations were established. For example, an iterative algorithm for solving the coupled matrix equations $\boldsymbol{AYB} = \boldsymbol{E}$, $\boldsymbol{CYD} = \boldsymbol{F}$ over generalized centro-symmetric matrices was proposed [39]. The coupled matrix equations

$$\begin{cases} \boldsymbol{AX} - \boldsymbol{YB} = \boldsymbol{E}, \\ \boldsymbol{CX} - \boldsymbol{YD} = \boldsymbol{F}, \end{cases}$$

were discussed in [40,41] and the conjugate gradient algorithms for its reflexive-solution and generalized reflexive-solution were given. The conjugate gradient squared and generalized product bi-conjugate gradient algorithms were suggested for two classes of coupled matrix equations respectively [42,43]. The conjugate gradient algorithms for the generalized bisymmetric solution and the reflexive solution of the coupled matrix equations

$$\begin{cases} \boldsymbol{AXB} + \boldsymbol{CYD} = \boldsymbol{M}, \\ \boldsymbol{EXF} + \boldsymbol{GYH} = \boldsymbol{N}, \end{cases}$$

were suggested in [44–46].

The generalized centrosymmetric solution of the coupled matrices

$$\begin{cases} \sum_{i=1}^{p} \boldsymbol{A}_i\boldsymbol{X}\boldsymbol{B}_i + \sum_{i=1}^{p} \boldsymbol{C}_i\boldsymbol{Y}\boldsymbol{D}_i = \boldsymbol{M}, \\ \sum_{i=1}^{p} \boldsymbol{E}_i\boldsymbol{X}\boldsymbol{F}_i + \sum_{i=1}^{p} \boldsymbol{G}_i\boldsymbol{Y}\boldsymbol{H}_i = \boldsymbol{N}, \end{cases}$$

was given by using the conjugate gradient algorithm [47]. To extend and generalize the above mentioned matrix equations, in this paper we consider the generalized coupled Sylvester matrix equations [13,48,49],

$$\begin{cases} \boldsymbol{A}_{11}\boldsymbol{X}_1\boldsymbol{B}_{11} + \boldsymbol{A}_{12}\boldsymbol{X}_2\boldsymbol{B}_{12} + \cdots + \boldsymbol{A}_{1q}\boldsymbol{X}_q\boldsymbol{B}_{1q} = \boldsymbol{F}_1, \\ \boldsymbol{A}_{21}\boldsymbol{X}_1\boldsymbol{B}_{21} + \boldsymbol{A}_{22}\boldsymbol{X}_2\boldsymbol{B}_{22} + \cdots + \boldsymbol{A}_{2q}\boldsymbol{X}_q\boldsymbol{B}_{2q} = \boldsymbol{F}_2, \\ \qquad\qquad\vdots \\ \boldsymbol{A}_{p1}\boldsymbol{X}_1\boldsymbol{B}_{p1} + \boldsymbol{A}_{p2}\boldsymbol{X}_2\boldsymbol{B}_{p2} + \cdots + \boldsymbol{A}_{pq}\boldsymbol{X}_q\boldsymbol{B}_{pq} = \boldsymbol{F}_p, \end{cases} \tag{1}$$

where $\boldsymbol{A}_{ij} \in \mathbb{R}^{r_i \times m_j}$, $\boldsymbol{B}_{ij} \in \mathbb{R}^{n_j \times s_i}$ and $\boldsymbol{F}_i \in \mathbb{R}^{r_i \times s_i}$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$, are known matrices, and $\boldsymbol{X}_j \in \mathbb{R}^{m_j \times n_j}$, $j = 1, 2, \ldots, q$ are unknown matrices to be determined. Except for the above mentioned coupled matrix equations, Eq. (1) includes some special matrix equations as its special cases. For example, if $p = 1$, then Eq. (1) becomes

$$\boldsymbol{A}_1\boldsymbol{X}_1\boldsymbol{B}_1 + \boldsymbol{A}_2\boldsymbol{X}_2\boldsymbol{B}_2 + \cdots + \boldsymbol{A}_q\boldsymbol{X}_q\boldsymbol{B}_q = \boldsymbol{F}, \tag{2}$$

where $\boldsymbol{F} \in \mathbb{R}^{r \times s}$, $\boldsymbol{A}_j \in \mathbb{R}^{r \times m_j}$ and $\boldsymbol{B}_j \in \mathbb{R}^{n_j \times s}$, $j = 1, 2, \ldots, q$, are known matrices, and $\boldsymbol{X}_j \in \mathbb{R}^{m_j \times n_j}$, $j = 1, 2, \ldots, q$, are unknown matrices to be determined. If Eq. (2) has only one unknown matrix, then it becomes

$$\boldsymbol{A}_1\boldsymbol{X}\boldsymbol{B}_1 + \boldsymbol{A}_2\boldsymbol{X}\boldsymbol{B}_2 + \cdots + \boldsymbol{A}_q\boldsymbol{X}\boldsymbol{B}_q = \boldsymbol{F}, \tag{3}$$

where $\boldsymbol{F} \in \mathbb{R}^{r \times s}$, $\boldsymbol{A}_j \in \mathbb{R}^{r \times m}$ and $\boldsymbol{B}_j \in \mathbb{R}^{n \times s}$, $j = 1, 2, \ldots, q$, are known matrices, and $\boldsymbol{X} \in \mathbb{R}^{m \times n}$ is unknown matrix to be determined. If $q = 1$, then Eq. (1) becomes

$$\begin{cases} \boldsymbol{A}_1\boldsymbol{X}\boldsymbol{B}_1 = \boldsymbol{F}_1, \\ \boldsymbol{A}_2\boldsymbol{X}\boldsymbol{B}_2 = \boldsymbol{F}_2, \\ \qquad\vdots \\ \boldsymbol{A}_p\boldsymbol{X}\boldsymbol{B}_p = \boldsymbol{F}_p, \end{cases} \tag{4}$$

where $\boldsymbol{A}_i \in \mathbb{R}^{r_i \times m}$, $\boldsymbol{B}_i \in \mathbb{R}^{n \times s_i}$ and $\boldsymbol{F}_i \in \mathbb{R}^{r_i \times s_i}$ are known matrices and $\boldsymbol{X} \in \mathbb{R}^{m \times n}$ is unknown to be determined.

Motivated and inspired by the above work, this paper discusses the conjugate gradient least squares algorithm for solving Eq. (1). We will prove that the iterative solution will converge to the exact solution when Eq. (1) is consistent and the iterative solution will converge to the least squares solution when Eq. (1) is inconsistent within the finite iteration steps in the absence of the roundoff errors. Also with the proper choice of the initial iterative matrix we will prove that the iterative solution will converge to the least squares and minimum-norm solution.

The remainder of this paper is organized as follows. Section 2 offers the symbols and the preliminaries. Section 3 brings the conjugate gradient least squares iterative algorithm and establishes several properties. A numerical example is offered in Section 4 to illustrate the effectiveness of the proposed algorithm. Finally, Some concluding remarks are given in Section 5.

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