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An inverse source problem for a two parameter anomalous diffusion equation with nonlocal boundary conditions

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ABSTRACT

We consider the inverse problem of determination of the solution and a source term for a time fractional diffusion equation in two dimensional space. The time fractional derivative is the Hilfer derivative. A bi-orthogonal system of functions in $L^2(\Omega)$, obtained from the associated non-self-adjoint spectral problem and its adjoint problem, is used to prove the existence and uniqueness of the solution of the inverse problem. The stability of the solution of the inverse problem on the given data is proved.

1. Introduction and preliminaries

We are concerned with the following diffusion equation

$$D_{0_+}^{\alpha,\gamma} u(x,y,t) - \Delta u(x,y,t) = f(x,y), \quad (x,y,t) \in Q_T := \Omega \times (0,T], \tag{1}$$

along with the initial and boundary conditions given by

$$I_{0_+}^{1-\gamma} u(x,y,t)|_{t=0} = \phi(x,y), \quad (x,y) \in \Omega := (0,1) \times (0,1), \tag{2}$$

$$u(0,y,t) = u(1,y,t),\ u_x(1,y,t) = 0, \quad (y,t) \in D := [0,1] \times [0,T], \tag{3}$$

$$u(x,0,t) = u(x,1,t) = 0, \quad (x,t) \in D, \tag{4}$$

where $D_{0_+}^{\alpha,\gamma}(\cdot)$ is the left sided fractional derivative of order α and type γ, introduced by Hilfer [1]:

$$D_{0_+}^{\alpha,\gamma} w(t) := \left[I_{0_+}^{(\gamma-\alpha)} \frac{d}{dt} \left(I_{0_+}^{(1-\gamma)} \right) \right] w(t), \quad 0 < \alpha \le \gamma < 1, \tag{5}$$

and the left sided fractional integral is defined by the formula

$$I_{0_+}^{\beta} w(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} w(\tau) d\tau, \quad t > 0,\ \beta > 0, \tag{6}$$

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where $\Gamma(\cdot)$ is the Euler gamma function, $f(x, y)$ is the source term, $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$, the two dimensional Laplacian and $\varphi(x, y)$ is the given initial temperature. The Laplace transform of the fractional derivative (5) is given by Hilfer in [1]

$$\mathcal{L}\{D_{0+}^{\alpha,\gamma} f(t)\} = s^{\alpha}\mathcal{L}\{f(t)\} - s^{\alpha-\gamma} I_{0+}^{1-\gamma} f(t)\Big|_{t=0}, \quad 0 < \alpha \le \gamma < 1. \tag{7}$$

Notice that the fractional derivative $D_{0+}^{\alpha,\gamma}$ reduces to the Riemann–Liouville fractional derivative and Caputo fractional derivative for $\gamma = \alpha$ and $\gamma = 1$, respectively,

$$D_{0+}^{\alpha,\alpha} w(t) := D_{0+}^{\alpha} w(t), \qquad D_{0+}^{\alpha,1} w(t) = D_{0+}^{\alpha}[w(t) - w(0)] := {}^{C}D_{0+}^{\alpha} w(t),$$

where $D_{0+}^{\alpha} w(t) = (d/dt) I_{0+}^{1-\alpha} w(t)$ and ${}^{C}D_{0+}^{\alpha} w(t) = I_{0+}^{1-\alpha}(d/dt) w(t)$ are the left sided Riemann–Liouville and Caputo fractional derivatives of order $0 < \alpha < 1$, respectively.

The determination of the solution $u(x, y, t)$ and the space dependent source term $f(x, y)$ for the problem (1) supplemented with the initial condition (2), the nonlocal boundary condition (3) and Dirichlet boundary condition (4) is the concern of this paper. We call this problem the inverse problem for the system (1)–(4).

The inverse problem is not solvable with the sole initial condition (2) and the boundary conditions (3)–(4); an extra condition is needed. We take

$$u(x, y, T) = \psi(x, y), \quad (x, y) \in \Omega. \tag{8}$$

We are looking for the map

$$\{\phi(x, y), u(x, y, T)\} \to \{f(x, y), u(x, y, t)\}, \quad t < T,$$

and want to know whether the map is one to one, i.e., the solution of the inverse source problem is unique?

By a regular solution of the inverse problem, we mean a pair of functions $\{u(x, y, t), f(x, y)\}$ such that $u(., ., t) \in C^2(\Omega, \mathbb{R})$, $D_{0+}^{\alpha,\gamma} u(x, y, .) \in C([0, T], \mathbb{R})$ and $f \in C(\Omega, \mathbb{R})$, which satisfy (1)–(4) and $u(x, y, T) = \psi(x, y)$. For the solution of the inverse problem, we use a bi-orthogonal system of functions constructed in [2]. It was used to prove the well-posedness of the direct and the inverse source problem for a time fractional diffusion equation involving Riemann–Liouville fractional derivative in [3]. The main purpose of this paper is to extend the results of [3] with a so called generalized fractional derivative as in Eq. (1); it is also generalization of the paper [4] to a two-dimensional space situation.

Fractional calculus has been considered by the scientists from all fields due to several applications in diverse fields from engineering, physics, finance to nanotechnology, see for example latest monographs and articles [1,5–11]. Several authors used the time/space (or both) fractional derivatives in the diffusion equation [9,12,13] to explain the nonstandard phenomena (anomalous diffusion) observed experimentally in diffusion process [14–17].

The time fractional diffusion equations with Riemann–Liouville or Caputo fractional derivatives are equivalent to infinitesimal generators of time fractional evolutions that arise in the transition from microscopic to macroscopic time scales [18]. The transition from first order derivative to the fractional order time derivative arises physically as reported by Hilfer [19–21]. The diffusion equation with Riemann–Liouville fractional derivative coincides with classical diffusion equation from both sides, that is, $\alpha \to 1_-$ and $\alpha \to 1_+$, where as diffusion equation with Caputo fractional derivative only coincides with classical diffusion equation with left side limit only [22]. It is desired to have properties of both of these fractional derivatives and the Hilfer fractional derivative provides this versatility in the solution of the dynamic equation involving Hilfer fractional derivative.

Let us dwell upon some recent articles dealing with inverse problems related to the time fractional diffusion equations. In [23] the direct and inverse problems involving Hadamard fractional derivative was considered for a parabolic equation in an appropriate Banach space. The initial condition of the type (2), i.e., the initial condition in terms of fractional integral was also considered in [24] where the author considered the inverse source problem and proved that solvability depends on the distribution of zeros of Mittag-Leffler type functions (for definition and some properties of Mittag-Leffler function see Section 2). An inverse source problem for a one dimensional time fractional diffusion with Hilfer fractional derivative was considered in [4]. For a nonlinear time fractional diffusion equation an inverse coefficient problem was considered in [25]. Wei et al. [26] proved unique identification of a space dependent source term for time fractional diffusion equation. An inverse source problem for space–time fractional differential equation is considered in [27]. The authors of [28] concluded that the nonlinear time fractional diffusion models are more suitable for simulating and depicting sub-diffusion process in porous medium under changing diffusion rate. In [29], a Tikhonov regularization algorithm is proposed to solve an inverse source problem for a space fractional diffusion equation in one space dimension. A multiple-scale radial basis function method is applied to solve direct and inverse Cauchy problem by Liu et al. [30]. An inverse source problem for one dimensional time fractional diffusion equation with zero Neumann boundary conditions is considered in [31]. Ali and Malik [32] considered direct and inverse source problem for time fractional diffusion equation with nonlocal boundary condition involving a positive parameter, both problems are proved to be well-posed in the sense of Hadamard. A topical review on inverse problems related to fractional diffusion equations is provided in [33].

The rest of the paper is organized as follows: in Section 2, we present some useful properties and estimates of the Mittag-Leffler function. We construct a bi-orthogonal system of functions in the same section. In Section 3, we prove the existence, uniqueness and stability results for the inverse problem.

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