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# Some uniqueness, multiplicity and complete dynamics for a cooperative model☆

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ABSTRACT

In this paper, a two-species cooperative model with diffusion and under homogeneous Dirichlet boundary conditions is investigated. It is shown the existence, stability, uniqueness and multiplicity of positive solutions. In particular, we study the global asymptotical stability of the unique positive solution when $a \in (\lambda_1(-\frac{b\theta_d}{1+\beta\theta_d}), \infty)$ and $\alpha$ is large. Our method of analysis is based on perturbation technique, the Lyapunov–Schmidt procedure and the bifurcation theory.



## 1. Introduction

In this paper, we study the following cooperative model

$$\begin{cases} \Delta u + u\left(a - u + \dfrac{bv}{(1+\alpha u)(1+\beta v)}\right) = 0, & x \in \Omega, \\ \Delta v + v\left(d - v + \dfrac{cu}{(1+\alpha u)(1+\beta v)}\right) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain in $R^n$ with smooth boundary $\partial\Omega$. $b, c, \alpha$ and $\beta$ are positive constants; both $a$ and $d$ may change sign.

If $\alpha = \beta = 0$, then (1.1) becomes the classical Lotka–Volterra cooperative model which has been extensively studied by many scholars, the interested readers may refer to [1–5] and the references therein. In particular, Korman and Leung [1] showed that if $a > \lambda_1$ and $d > \lambda_1$, then (1.1) has a pair of positive solutions if and only if $bc < 1$. Moreover, in [4], they studied that if $a > \lambda_1(b\theta_d)$, $d > \lambda_1(c\theta_a)$ and $bc < 1$, then (1.1) has at least a pair of positive solutions. For the case $a < \lambda_1$ and $d < \lambda_1$, Lou [3] proved that if $bc \le 1$, then (1.1) has no positive solution. Furthermore, when the space dimension $n \le 5$, (1.1) has a pair of positive solutions if and only if $bc > 1$; when $n \ge 6$ and $\Omega$ is star-shaped, then (1.1) has no positive solution provided $a = d \le (n - \frac{6}{n})\lambda_1$.

The purpose of this paper is mainly concerned with the qualitative property of positive solutions of (1.1) with $\alpha > 0$ and $\beta > 0$. By making use of the local bifurcation theory of Rabinowitz [6], we can see that $(a; u, v) = (\lambda_1(-\frac{b\theta_d}{1+\beta\theta_d}); 0, \theta_d)$ is

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a simple bifurcation point. In fact, from the global bifurcation point of view, the local bifurcation solution can be extended to $\infty$ by increasing $a$ to $\infty$. Moreover, if $a'(0) < 0$, then there must be a turning point $a^*$ such that $a^* < \lambda_1(-\frac{b\theta_d}{1+\beta\theta_d})$. If $a \in (a^*, \lambda_1(-\frac{b\theta_d}{1+\beta\theta_d}))$, then (1.1) has at least two positive solutions. Furthermore, if every turning point turns to right, then (1.1) has exactly two positive solutions when $a \in (a^*, \lambda_1(-\frac{b\theta_d}{1+\beta\theta_d}))$.

However, we find that for large $\alpha$ with fixed $d > \lambda_1$, when $a$ falls into different ranges, (1.1) has different type of positive solutions. In fact, one can use the regular or singular perturbation techniques to obtain a good understanding of uniqueness for (1.1). In particular, we can claim that if $a \in (\lambda_1(-\frac{b\theta_d}{1+\beta\theta_d}), \infty)$, then (1.1) has a unique positive solution for large $\alpha$. Moreover, we can show that the unique positive solution is asymptotically stable. Furthermore, by the theory of monotone dynamical systems [7], we can also claim that this unique positive solution is globally asymptotically stable. In fact, when $a$ is large and $\alpha$ is bounded, we can also claim that (1.1) has a unique positive solution, which is globally asymptotically stable.

Finally, by virtue of the Lyapunov–Schmidt technique, more information about the branch of positive solutions of (1.1) near $(a, d) = (\lambda_1, \lambda_1)$ is obtained. In particular, we can prove that if $t \in (0, \pi/2)$ sufficiently close to $\pi/2$, then (1.1) has a branch bifurcating $(0, 0)$ can reach $(0, \theta_d)$. Similarly, if $t \in (0, \pi/2)$ sufficiently close to 0, then (1.1) has a branch bifurcating $(0, 0)$ can reach $(\theta_a, 0)$. Moreover, we can always find some $b, c$ such that any positive solution of (1.1) is unstable for $a$ and $d$ sufficiently close to $\lambda_1$, respectively. Meanwhile, we can also find some $b, c$ such that any positive solution of (1.1) is linearly stable.

We introduce some notations and basic facts. Let $\lambda_1(p) < \lambda_2(p) \le \lambda_3(p) \le \cdots$ be all eigenvalues of the following problem

$$-\Delta u + p(x)u = \lambda u, \qquad u|_{\partial\Omega} = 0,$$

where $p \in C^\sigma(\bar{\Omega})$. It is well known that $\lambda_1(p)$ is simple, real and $\lambda_1(p)$ is strictly increasing in the sense that if $p_1 \le\not\equiv p_2$ implies that $\lambda_1(p_1) < \lambda_1(p_2)$. When $p \equiv 0$, we denote $\lambda_1(0)$ by $\lambda_1$. Moreover, we denote by $\phi_1$ the positive eigenfunction corresponding to $\lambda_1$ with normalization $\|\phi_1\|_\infty = 1$.

It is well known that for any $a > \lambda_1$, the problem

$$-\Delta u = u(a - u), \qquad u|_{\partial\Omega} = 0 \tag{1.2}$$

has a unique positive solution which we denote by $\theta_a$. It is also known that $\theta_a$ is continuously differentiable, strictly increasing in $(\lambda_1, +\infty)$. Moreover, it is non-degenerate and linearly stable.

The rest of this paper is organized as follows. In Section 2, we give some preliminary results which are needed in the later sections. In Section 3, we establish the existence of positive solutions by the bifurcation theory. In Section 4, we study the existence and the uniqueness of positive solution for large $\alpha$. Finally, we further study existence and stability of positive solutions near $(a, d) = (\lambda_1, \lambda_1)$ in Section 5.

## 2. Preliminaries

In this section, we state some notations and basic facts which will be used in this paper.

For $p > N$, we define two Banach spaces $X$ and $Y$ as

$$\begin{cases} X = [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)] \times [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)], \\ Y = L^p(\Omega) \times L^p(\Omega). \end{cases}$$

Then the Sobolev embedding theory deduces that $X \subseteq C^1(\bar{\Omega}) \times C^1(\bar{\Omega})$.

By virtue of the maximum principle, we first give a priori estimates, but we omit the proof here.

**Lemma 2.1.** *Let $(u, v)$ be any positive solution of (1.1). Then we have*

$$u \le a + \frac{b}{\beta}, \qquad v \le d + \frac{c}{\alpha}. \tag{2.1}$$

Due to the monotonicity of two-species cooperative systems, we have the following well-known results (see [7]):

(a) If there is no coexistence state, then one of the semi-trivial equilibria is unstable and the other one is globally asymptotically stable.
(b) If there is a unique coexistence state and it is stable, then it is globally asymptotically stable.
(c) If all coexistence states are asymptotically stable, then there is a unique coexistence state, which is globally asymptotically stable.

By the theory of monotone dynamical systems, we can get the following result.

**Lemma 2.2.** *If $a > \lambda_1$ and $d > \lambda_1$, then (1.1) has at least a stable coexistence state.*

From the monotone method, we have the following lemma.