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# An improved non-traditional finite element formulation for solving three-dimensional elliptic interface problems

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ABSTRACT

Solving elliptic equations with interfaces has wide applications in engineering and science. The real world problems are mostly in three dimensions, while an efficient and accurate solver is a challenge. Some existing methods that work well in two dimensions are too complicated to be generalized to three dimensions. Although traditional finite element method using body-fitted grid is well-established, the expensive cost of mesh generation is an issue. In this paper, an efficient non-traditional finite element method with non-body-fitted grids is proposed to solve elliptic interface problems. The special cases when the interface cuts though grid points are handled carefully, rather than perturbing the cutting point away to apply the method for general case. Both Dirichlet and Neumann boundary conditions are considered. Numerical experiments show that this method is approximately second order accurate in the $L^\infty$ norm and $L^2$ norm for piecewise smooth solutions. The large sparse matrix for our linear system also has nice structure and properties.


## 1. Introduction

In recent years, there has been tremendous interest in developing efficient numerical methods for interface problems. In numerical mathematics, ordinary differential equations or partial differential equations are often used to model interface problems involving materials with different properties, such as conductivities, densities or diffusions. Elliptic problem with internal interfaces is the basic form of this kind of problem, the governing equations of which has discontinuous coefficients at interfaces and sometimes singular source term exists. Due to the poor global smoothness of the solution and the irregularity of the interfaces, the solution for the differential equation model is usually not smooth and even discontinuous. In this way, standard finite element or finite difference method are not suitable for interface problems, hence designing highly efficient methods for these problems are desired. To solve this problem, a large number of numerical methods are designed for interface problems, which can be classified into three main categories, including finite difference methods, finite element methods and finite volume methods.

The numerical model of interface problem was first proposed by Peskin [1] in 1977 to study blood flow through heart valves. The immersed boundary method (IBM) [2] is a mathematical formulation and numerical approach to such problems. This method uses a numerical approximation of the delta-function, which smears out the solution on a thin finite band around the interface gama. In the following years, Peskin focused on the improvement of the immersed boundary method

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and several adaptive version of the method are introduced [3–6], actual second order convergence rates are obtained by considering the interaction of a viscous incompressible flow and an anisotropic incompressible viscoelastic shell. In 1994, LeVeque and Li [7–9] proposed the immersed interface method (IIM) to solve elliptic equations with discontinuous coefficients and singular source term. This method is based on the finite difference method under the Cartesian grid. Standard finite difference or finite element method is employed away from the interface, while the grid points or elements near the interface are amended by the interface condition. This method is successfully applied to incompressible Stokes equation and Navier–Stokes equations with singular source term. Liu, Fedkiw and Kang proposed the ghost fluid method (GFM) to solve the elliptic equations with interfaces in 2000 [10,11], which is widely used in solving incompressible two-phase or multiphase flow problem, interface tracking problem and irregular domain problems with Dirichlet boundary condition. Wei et al. pays attention to interface problems with geometry singularity since 2007 and developed the second order accurate method called matched interface and boundary method (MIB) [12–14]. This method is successfully applied to biomathematics research on molecular level.

So far, there are mainly two kinds of effective finite element methods, one is fitted interface finite element method, in which meshes are generated along the interface. The other is immersed interface finite element method, in which the mesh generation does not rely on the interface, while the construction of finite element space relies on the jump condition of the interface. As for fitted mesh method, Chen and Zou [15] considered the finite element methods for solving second order elliptic and parabolic interface problems. The fitted methods are computationally costly especially for problems with moving interfaces because repeated grid generation is called for. Examples of immersed interface method are immersed finite element method [16–23], adaptive immersed interface method [24], extended finite element method [25–27]. The penalty finite element method [28,29] modifies the bilinear form near the interface by penalizing the jump of the solution value (with no general flux jump) across the interface. Recently a few unfitted mesh methods based on discontinuous Galerkin method using well known interior penalty technique to deal with jump and flux conditions for elliptic interface problem [30–32].

Also, there has been a large body of work from the finite volume perspective for developing high order methods for elliptic equations in complex domains, such as [33,34] for two-dimensional problems and [35] for three-dimensional problems. In [36], the immersed finite volume element method (IFVE) is developed by combining the finite volume element method and the immersed finite element method.

Non-traditional finite element method has been developed for solving interface problems involving several important types of partial differential equations, such as elliptic equations [37–40], elliptic interface problems with triple junction points [41] and elasticity interface problems [42,43]. The main idea is that the test function and trial function have different basis and the resulting linear system is non-symmetric but positive definite. Although the three dimensional elliptic interface problem was discussed in [39] using this type of formulation, whenever the interface hit grid points, it is perturbed away.

In this paper, we propose an improved numerical method for solving the three-dimensional elliptic interface problem with Dirichlet Boundary Condition and Neumann Boundary Condition. We discussed all the possible ways the interface cuts the grid. In other words, the interface is not perturbed away if it hits grid points. Matrix coefficients $\beta^+$ and $\beta^-$ can be handled. Due to the weak formulation, the case with second derivative blowing up at a point can be handled as well. Extensive numerical experiments demonstrate that our new method is effective for all kinds of possible problems and can achieve second order accuracy in the $L^\infty$ norm and $L^2$ norm.

## 2. Equations and weak formulation

Consider an open bounded domain $\Omega \subset R^3$. Let $\Gamma$ be an interface of co-dimension 1, which divides $\Omega$ into disjoint open subdomains, $\Omega^-$ and $\Omega^+$, hence $\Omega = \Omega^- \bigcup \Omega^+ \bigcup \Gamma$. Assume that the boundary $\partial\Omega$ and the boundary of each subdomain $\partial\Omega^\pm$ are Lipschitz continuous as submanifolds. Since $\partial\Omega^\pm$ are Lipschitz continuous, so is $\Gamma$. A unit normal vector of $\Gamma$ can be defined a.e. on $\Gamma$.

We seek solutions of the variable coefficient elliptic equation away from the interface $\Gamma$ given by

$$-\bigtriangledown \cdot (\beta(\mathbf{x}) \bigtriangledown u(\mathbf{x})) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega \setminus \Gamma \tag{1}$$

in which $\mathbf{x} = (x_1, x_2, x_3)$ denotes the spatial variables and $\bigtriangledown$ is the gradient operator. The coefficient $\beta(\mathbf{x})$ is assumed to be a $3 \times 3$ matrix that is uniformly elliptic on each disjoint subdomain, $\Omega^-$ and $\Omega^+$, and its components are continuously differentiable on each disjoint subdomain, but they may be discontinuous across the interface $\Gamma$. The right-hand side $f(\mathbf{x})$ is assumed to lie in $L^2(\Omega)$.

Given functions $a$ and $b$ along the interface $\Gamma$, we prescribe the jump conditions

$$\begin{cases} [u]_\Gamma(\mathbf{x}) \equiv u^+(\mathbf{x}) - u^-(x) = a(\mathbf{x}) \\ [(\beta \bigtriangledown u) \cdot \mathbf{n}]_\Gamma(\mathbf{x}) \equiv \mathbf{n} \cdot (\beta^+(\mathbf{x}) \bigtriangledown u^+(\mathbf{x})) - \mathbf{n} \cdot (\beta^-(\mathbf{x}) \bigtriangledown u^-(\mathbf{x})) = b(\mathbf{x}). \end{cases} \tag{2}$$

The "$\pm$" superscripts refer to limits taken from within the subdomains $\Omega^\pm$.

Finally, the boundary conditions are given by

$$u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{3}$$

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